$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$$\int_{-\infty}^{\infty} \delta(t-a) f(t)\,dt$$

Example

Given

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

prove that

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{+j\omega t}\, d\omega$$

---

define $$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

where $$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(a) e^{-j\omega a}\, da \right] e^{j\omega t}\, d\omega$$

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(a) \underbrace{\left[\int_{-\infty}^{\infty} e^{j\omega(t-a)}\, d\omega \right]}_{2\pi\,\delta(t-a)} da$$

$$y(t) = \frac{2\pi}{2\pi} \int_{-\infty}^{\infty} f(a)\, \delta(t-a)\, da$$

$$\boxed{y(t) = f(t)}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

Example

Real function in time
have conjugate sym. spectra

given $f(t)$ is real

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cos(\omega t) dt - j \int_{-\infty}^{\infty} f(t) \sin(\omega t) dt$$

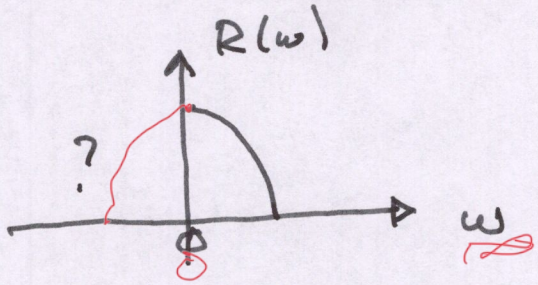
$$F(\omega) = R(\omega) - j X(\omega)$$

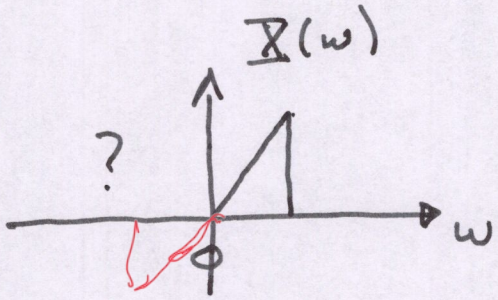
$$R(\omega) = \int_{-\infty}^{\infty} f(t) \cos(\omega t) dt$$

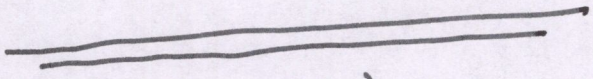
$$X(\omega) = \int_{-\infty}^{\infty} f(t) \sin(\omega t) dt$$

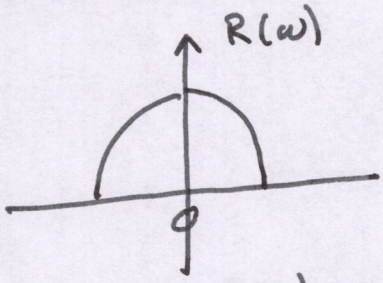
(1) $R(\omega) = R(-\omega)$

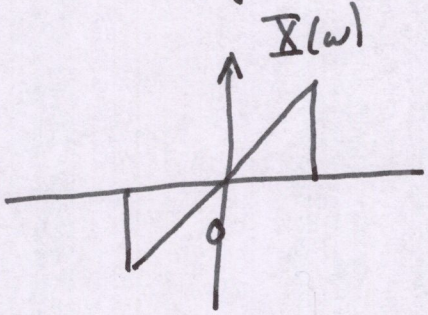
(2) $X(\omega) = -X(-\omega)$


R(ω)
?
ω
0


X(ω)
?
ω
0




R(ω)
0


X(ω)
0

Parseval's Thm

$$\int_{-\infty}^{\infty} |f|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

$f \longleftrightarrow F(\omega)$

$f^* \longleftrightarrow ?$

$$\Rightarrow F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$F^*(\omega) = \int_{-\infty}^{\infty} f^* e^{j\omega t} dt$$

$$F^*(-\omega) = \int_{-\infty}^{\infty} f^* e^{-j\omega t} dt \Longrightarrow f^* \longleftrightarrow F^*(-\omega)$$

$$ff^* \longleftrightarrow \frac{F(\omega) * F^*(-\omega)}{2\pi}$$

$$\int_{-\infty}^{\infty} |f|^2 e^{-j\omega t} dt = \int_{-\infty}^{\infty} \frac{F(z) F^*[-(\omega - z)]}{2\pi} dz$$

$\omega \equiv 0$

$$\int_{-\infty}^{\infty} |f|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(z) F^*(z) dz$$

$$\Rightarrow \int_{-\infty}^{\infty} |f|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(z)|^2 dz$$

Correlation

$$R(t) = \int_{-\infty}^{\infty} x(t+\tau)\, h(\tau)\, d\tau$$

$R(t) = x \circledast h$ cross-cor.

$\mathcal{F}(R(t)) = S_{xh}(\omega)$ cross-spectrum

$$S_{xh}(\omega) = \int_{-\infty}^{\infty} R(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t+\tau) h(\tau) \, d\tau \right] e^{-j\omega t} dt$$

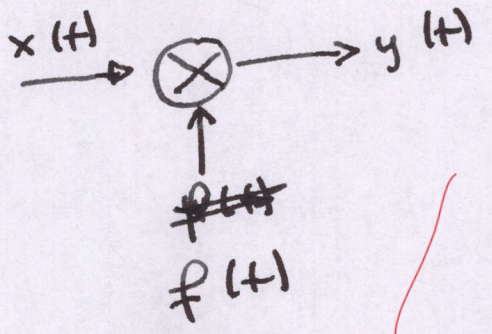
$$= \int_{-\infty}^{\infty} h(\tau) \left[\underbrace{\int_{-\infty}^{\infty} x(t+\tau) e^{-j\omega t} dt}_{\substack{t+\tau = \alpha \\ dt = d\alpha}} \right] d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau) \left[\underbrace{\int_{-\infty}^{\infty} x(\alpha) e^{-j\omega\alpha} d\alpha}_{\overline{X}(\omega)} \right] e^{j\omega\tau} d\tau$$

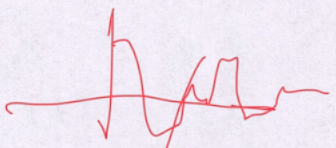
$$\boxed{S_{xh}(\omega) = \overline{X}(\omega) H(-\omega)}$$

$$\boxed{x \circledast h \longleftrightarrow \overline{X}(\omega) H(-\omega)}$$

Sampling & Reconstruction



$\cos(\omega t)\, u(t)$



where $f(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)\, T_s$

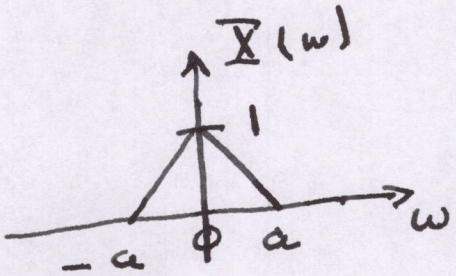
$FS(f) \Rightarrow f(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{j\omega_0 n t}$

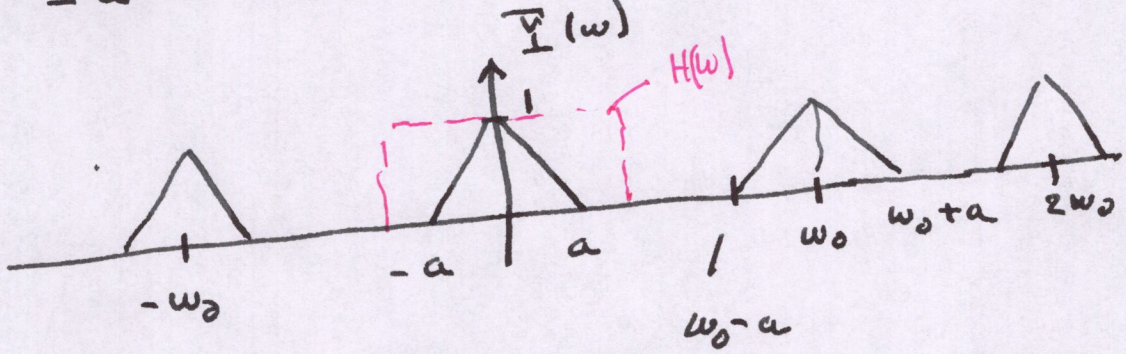
where $\omega_0 = \frac{2\pi}{T_s}$

$$y(t) = x(t) \sum_{n=-\infty}^{\infty} e^{j\omega_0 n t}$$

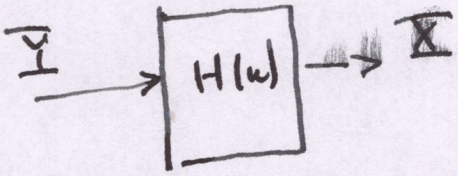
$$Y(\omega) = \frac{X(\omega) * \sum_{n=-\infty}^{\infty} \delta(\omega - \omega_0 n)\, 2\pi}{2\pi}$$

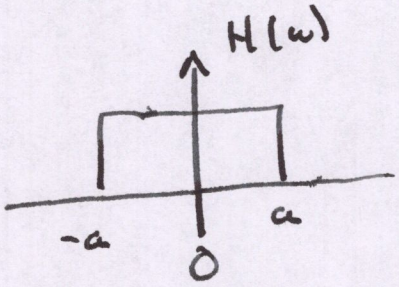
$$\boxed{Y(\omega) = \sum_{n=-\infty}^{\infty} X(\omega - n\omega_0)}$$





Reconstruction





ideal Low-pass filter

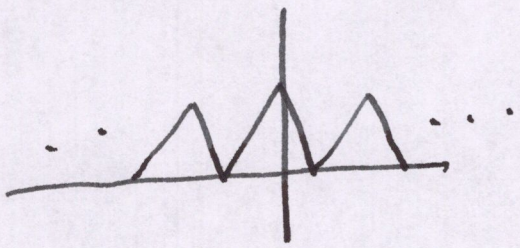
given that

$w_0 - a > a$

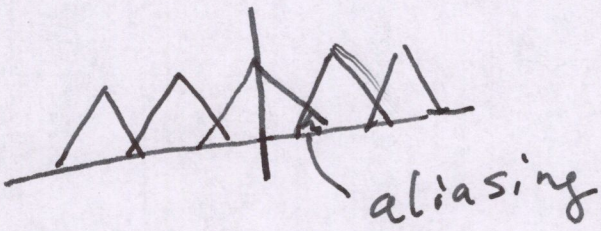
$\boxed{w_0 > 2a}$

Nyquist freq

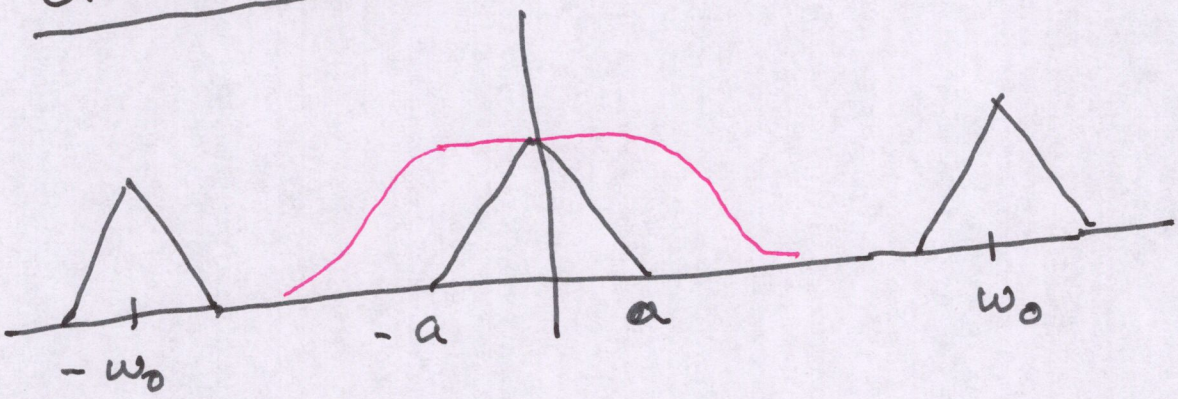
$\omega_0 = 2a$



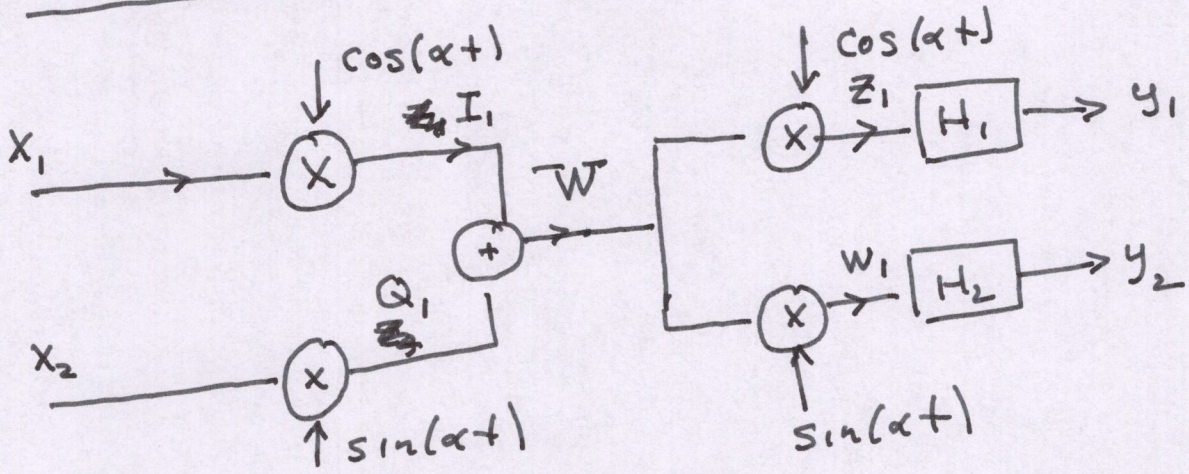
$\omega_0 < 2a$



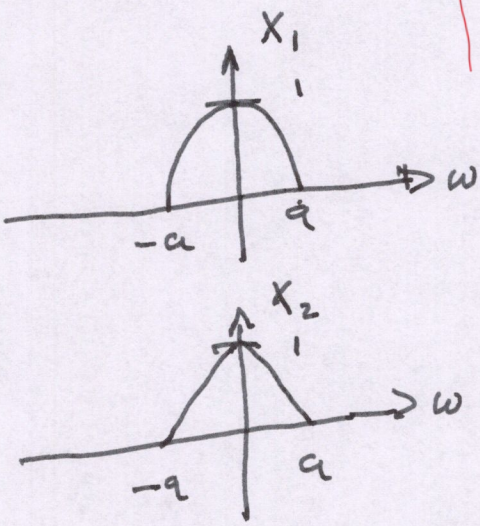
oversampling



Example



where



① $\mathcal{F}(\cos(\alpha t)) = \pi\left[\delta(\omega-\alpha)+\delta(\omega+\alpha)\right]$

② $\mathcal{F}(\sin(\alpha t)) = -i\pi\left[\delta(\omega-\alpha)-\delta(\omega+\alpha)\right]$

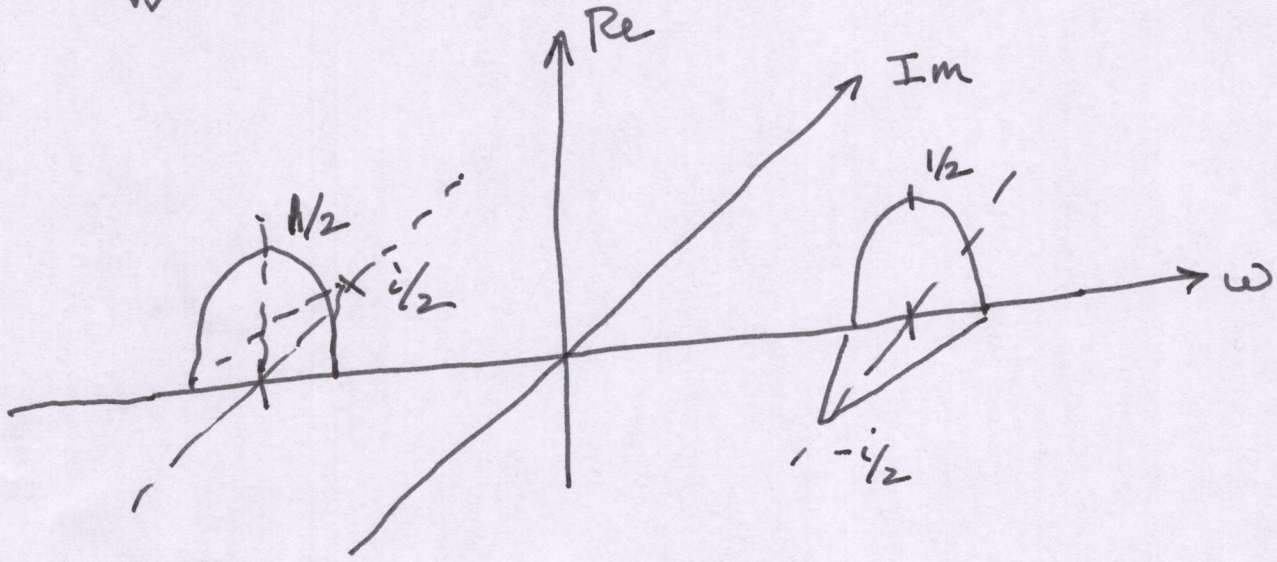
$$I_1 = \frac{1}{2\pi}\, X_1 * \mathcal{F}(\cos(\alpha t))$$

$$Q_1 = \frac{1}{2\pi}\, X_2 * \mathcal{F}(\sin(\alpha t))$$

$$I_1 = \frac{1}{2}\left[X_1(\omega-\alpha) + X_1(\omega+\alpha)\right]$$

$$Q_1 = \frac{-i}{2}\left[X_2(\omega-\alpha) - X_2(\omega+\alpha)\right]$$

$$W = I_1 + Q_1$$



$$Z_1 = \frac{\bar{W}}{2\pi} * \pi\left[\delta(\omega-\alpha) + \delta(\omega+\alpha)\right]$$

$$Z_2 = \frac{\bar{W}}{2\pi} * \pi(-i)\left[\delta(\omega-\alpha) - \delta(\omega+\alpha)\right]$$

$$Z_1 = \frac{1}{2}\left[\bar{W}(\omega-\alpha) + W(\omega+\alpha)\right]$$

$$Z_2 = \frac{(-i)}{2}\left[W(\omega-\alpha) - \bar{W}(\omega+\alpha)\right]$$

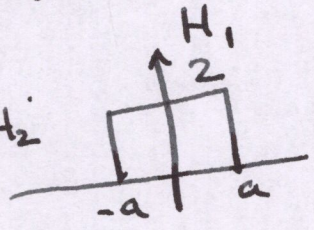
---

where $W(\omega) = \frac{1}{2}\left[\bar{X}_1(\omega-\alpha) + \bar{X}_1(\omega+\alpha)\right]$
$-\frac{i}{2}\left[\bar{X}_2(\omega-\alpha) - \bar{X}_2(\omega+\alpha)\right]$

$$Z_1 = \bar{X}_1(\omega)\frac{1}{2} + \text{HOT}$$

$$Z_2 = \bar{X}_2(\omega)\frac{1}{2} + \text{H.O.T}$$

$H_1 = H_2$



Example

$$y' + 6y = \delta(t)$$

find causal solution

given $y(0) = a$

① $\overline{Y}(\omega) = \int_{0^-}^{\infty} y(t) e^{-j\omega t} dt$

② $\int_{0^-}^{\infty} \frac{dy}{dt} e^{-j\omega t} dt = y e^{-j\omega t}\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} (-j\omega) y e^{-j\omega t} dt$

$= -y(0^-) + j\omega \overline{Y}$

③ $\int_{0^-}^{\infty} \delta(t) e^{-j\omega t} dt = 1$

$\int_{0^-}^{\infty} y' e^{-j\omega t} dt + \int_{0^-}^{\infty} 6y e^{-j\omega t} dt = \int_{0^-}^{\infty} \delta(t) e^{-j\omega t} dt$

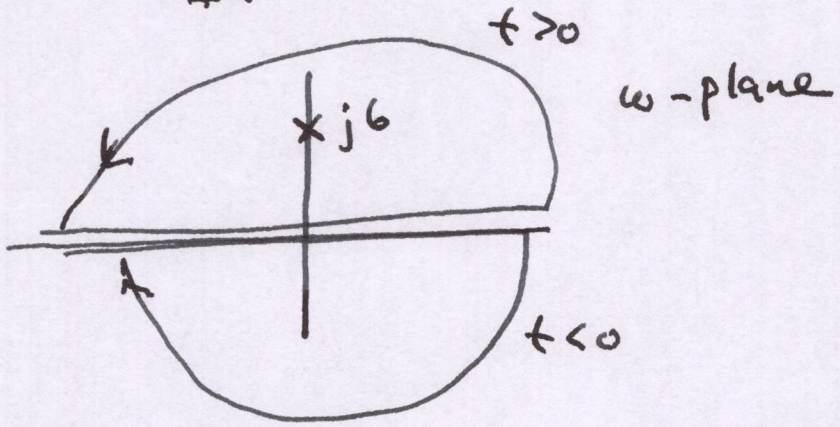
$-y(0^-) + j\omega \overline{Y} + 6\overline{Y} = 1$

$\overline{Y}(j\omega + 6) = 1 + y(0^-)$

$$\overline{Y} = \frac{1}{j\omega + 6} + \frac{y(0^-)}{j\omega + 6}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega) e^{j\omega t} d\omega = y(t)$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1+y(0^-)}{j(\omega - j6)} e^{j\omega t} d\omega$$



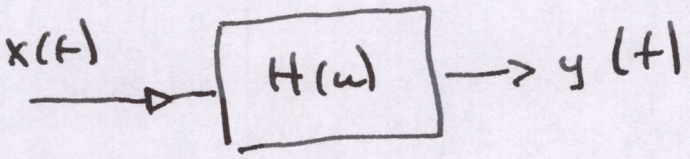
$t \geq 0$

$$y = \frac{1}{2\pi}\left[\frac{1+y(0^-)}{j(\omega - j6)} e^{j\omega t}\right](\omega - j6)2\pi j \Big|_{\omega = j6}$$

$$y = e^{-6t}(1 + y(0^-))$$

$t < 0$

$$y = 0$$



$$x(t) = e^{6t}u(-t)$$

$$H(\omega) = \frac{1}{(\omega+j6)(\omega-j6)}$$

find $y(t)$