Problem

$$y(n) = \frac{1}{2} y(n-1) + x(n)$$

IC: $y(-1) = a$

source: $x(n) = \alpha^n u(n)$

Find $y(n)$ for $n \geq 0$

$x(t) = \delta(t)$

$y(t) = \delta(t-a)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

~~$x(t) = \delta(t)$~~ $\Leftrightarrow$ =

~~$x[t] =$~~ $\delta(x)$

soln

$$\sum_{n=0}^{\infty} y[n] z^{-n} = \frac{1}{2} \sum_{n=0}^{\infty} y[n-1] z^{-n} + \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$Y(z) = \sum_{n=0}^{\infty} y[n] z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$Y(z) = \frac{1}{2} \sum_{n=0}^{\infty} y[n-1] z^{-n} + X(z)$$

$m = n-1$
or
$n = m-1$

$$Y(z) = \frac{1}{2} \sum_{m=-1}^{\infty} y[m] z^{-m} z^{-1} + X(z)$$

$$Y(z) = \frac{1}{2} y[-1] + \frac{1}{2} \sum_{m=0}^{\infty} y[m] z^{-m} z^{-1} + X(z)$$

$$Y(z) = \frac{1}{2} y[-1] + \frac{1}{2} Y z^{-1} + X(z)$$

$$Y(z) - \frac{1}{2} z^{-1} Y = \frac{1}{2} y[-1] + X(z)$$

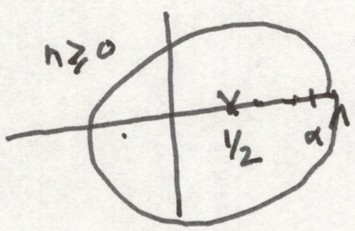
$$\boxed{Y(z) = \frac{\frac{1}{2} y[-1]}{1 - \frac{1}{2} z^{-1}} + \frac{X(z)}{1 - \frac{1}{2} z^{-1}}}$$

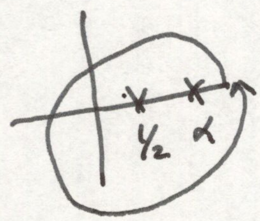
$$X(z) = \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n = \frac{1}{1-\frac{\alpha}{z}} \qquad |z| > |\alpha|$$

$$Y(z) = \frac{\frac{1}{2} y[-1]}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1-\frac{\alpha}{z}} \frac{1}{1-\frac{1}{2} z^{-1}}$$

$$ROC_y \quad |z| > \tfrac{1}{2} \cap |z| > |\alpha|$$

$$Y(z) = \underbrace{\frac{\frac{1}{2} y[-1] z}{z - 1/2}}_{Y_1} + \underbrace{\frac{z^2}{(z-\alpha)(z-1/2)}}_{Y_2}$$





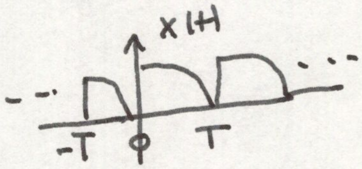
$(n+1) \geq 0$

$$y_1[n] = \frac{1}{2} y[-1] \left(\frac{1}{2}\right)^n$$

$$y_2[n] = \frac{(\alpha)^{n+1}}{\alpha - \frac{1}{2}} + \frac{\left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2} - \alpha}$$

Fourier Series

Given: $x(t) = x(t + kT) \qquad k = \{\ldots -1, 0, 1, 2\}$



$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{+j\omega_0 n t}$$

where $\omega_0 = \frac{2\pi}{T}$

Finding a_n

$$e^{-j\omega_0 k t} x(t) = e^{-j\omega_0 k t} \sum_{n=-\infty}^{\infty} a_n e^{j\omega_0 n t}$$

$$\int_0^T e^{-j\omega_0 k t} x(t)\, dt = \sum_{n=-\infty}^{\infty} a_n \int_0^T e^{j\omega_0 (n-k) t}\, dt$$

$$\int_0^T e^{j\omega_0 (n-k)}\, dt = \begin{cases} T & n-k=0 \\ 0 & n-k \neq 0 \end{cases}$$

therefore

$$a_n = \frac{1}{T}\int_0^T x(t)\, e^{j\omega_0 n t}$$

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generalized

$\phi_n(t)$ base function

$$f(t) = \sum_{n=-\infty}^{\infty} a_n\, \phi_n(t)$$

$$\phi_m^* f(t) = \sum_{n=-\infty}^{\infty} a_n\, \phi_n\, \phi_m^*$$

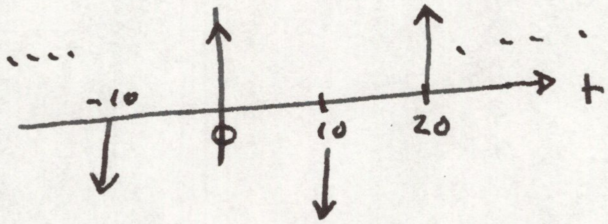
$$\int_0^T \phi_m^* f(t)\, dt = \sum_{n=-\infty}^{\infty} a_n \int_0^T \phi_n\, \phi_m^*\, dt$$

if $\int_0^T \phi_n\, \phi_m^*\, dt = \begin{cases} \lambda_n & n = m \\ 0 & n \neq m \end{cases}$

ϕ_m^*, ϕ_n are orthogonal

example

$$x(t) = \sum_{k=-\infty}^{\infty} (-1)^k \delta(t - k10)$$

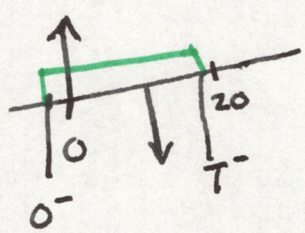


period $T = 20$

fund. freq $w_0 = \frac{2\pi}{T} = \frac{2\pi}{20}$

$$a_n = \frac{1}{T}\int_{0^-}^{T^-} \left[\quad ? \quad \right] e^{-jw_0 n t}\, dt$$

$$\Uparrow$$



$$a_n = \frac{1}{T}\int_{0^-}^{T^-} \left[\delta(t) - \delta(t-10)\right] e^{-jw_0 n t}\, dt$$

$$a_n = \frac{1}{T}\left[1 - e^{-jw_0 n 10}\right]$$

Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Special function

Delta function $\delta(t)$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$X(\omega) = 1$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t}\, d\omega = \delta(t)$$

$$\boxed{\delta(t) \overset{F}{\longleftrightarrow} 1}$$

$$\boxed{\int_{-\infty}^{\infty} e^{j\omega t}\, \cancel{dt}\, d\omega = 2\pi\, \delta(t)}$$

Step function

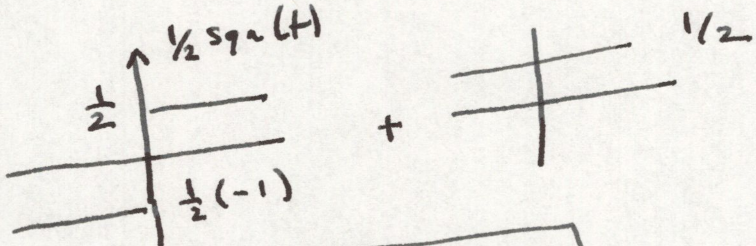
$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases}$$

$x(t) = u(t)$

$$X(\omega) = \int_0^\infty e^{-j\omega t}\,dt$$

$$= \left.\frac{e^{-j\omega t}}{-j\omega}\right|_0^\infty = ?$$

$x(t) = \frac{1}{2}\,\text{sign}(t) + \frac{1}{2}$



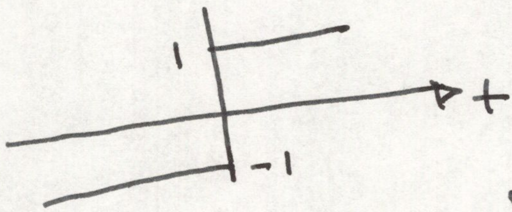
$$\boxed{\frac{1}{2} \overset{F}{\longleftrightarrow} 2\pi\,\delta(\omega)\,\frac{1}{2}}$$

recall $\int_{-\infty}^{\infty} e^{j\omega t}\,d\omega = 2\pi\,\delta(t)$

then $\int_{-\infty}^{\infty} e^{-j\omega t}\,dt = 2\pi\,\delta(\omega)$

note $\int_{-\infty}^{\infty} e^{\pm j\omega t}\,dt = 2\pi\,\delta(\omega)$

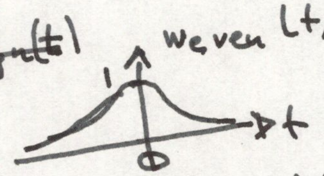
$\mathcal{F}(\text{sgn}(t))$



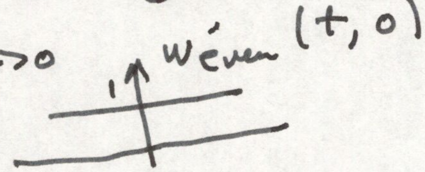
① $\int_{-\infty}^{\infty} \text{sgn}(t)\,dt = 0 \Leftarrow \underset{T}{\overset{\infty}{\uparrow}} \int_{-T}^{T} \text{sgn}(t)\,dt$

② replace $\text{sgn}(t)$ with $W_{even}(t,\epsilon)\,\text{sgn}(t)$

~~$w_{even}(t)\,\text{sgn}(t)$~~



such that $\epsilon \to 0$



$$\mathcal{F}\{\text{sgn}(t)\} = \underset{\epsilon}{\overset{0}{\uparrow}} \cdot e^{-\epsilon|t|}\,\text{sgn}(t)$$

③ $$\mathcal{F}(\text{sgn}(t)) = \underset{\epsilon}{\overset{0}{\uparrow}} \int_{-\infty}^{\infty} e^{-\epsilon|t|}\,\text{sgn}(t)\,e^{-j\omega t}\,dt$$

$$= \underset{\epsilon}{\overset{0}{\uparrow}} \left[\frac{1}{i\omega+\epsilon} + \frac{1}{i\omega-\epsilon}\right]$$

$$\boxed{\mathcal{F}(\text{sgn}(t)) = \frac{2}{i\omega}}$$

$$u(t) = \frac{1}{2}\,\mathrm{sgn}(t) + \frac{1}{2}$$

$$\mathcal{F}(u(t)) = \mathcal{F}\left(\frac{1}{2}\,\mathrm{sgn}(t)\right) + \mathcal{F}\left(\frac{1}{2}\right)$$

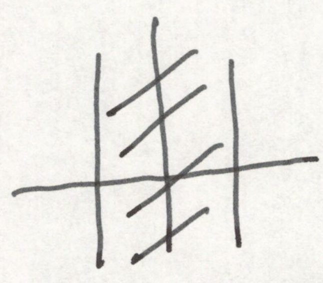
$$= \frac{1}{2}\,\frac{2}{i\omega} + \pi\delta(\omega)$$

$$\boxed{u(t) \longleftrightarrow \frac{1}{i\omega} + \pi\delta(\omega)}$$

$$\mathcal{L}(x(t))\Big|_{s=j\omega} = \mathcal{F}(x(t))$$

if $\mathcal{L}(x(t))$ ROC includes the $j\omega$-axis

s-plane

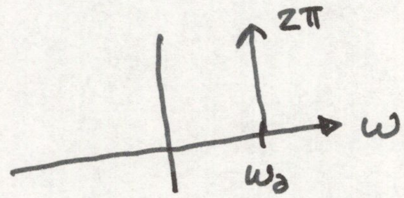


The complex exp

A. $$\boxed{e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\,\delta(\omega-\omega_0)}$$

proof
$$\int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t}\,dt$$

$$\mathcal{F}(e^{j\omega_0 t}) = 2\pi\,\delta(\omega-\omega_0)$$



B. Modulation

$$\boxed{x(t)\,e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega-\omega_0)}$$

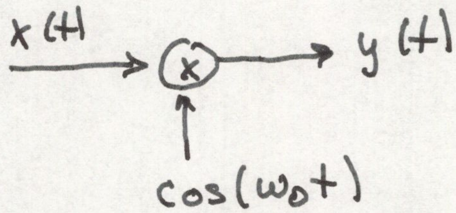
proof
given $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\,dt$

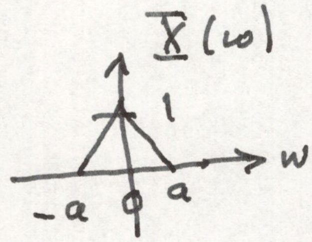
$$? = \int_{-\infty}^{\infty} x(t) e^{+j\omega_0 t} e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-j(\omega-\omega_0)t}\,dt$$

$$X(\omega) = X(\omega-\omega_0)$$

Example



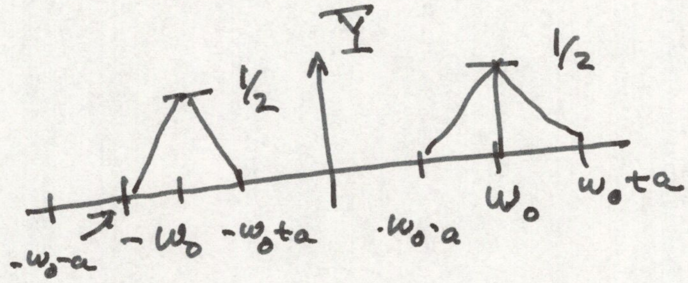


Find $Y(\omega)$

$$y(t) = x(t)\cos(\omega_0 t)$$

$$= \frac{x(t)}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

$$\mathcal{F}(y(t)) = Y(\omega) = \cancel{\frac{1}{2\pi}}\,\frac{1}{2}\left[X(\omega-\omega_0) + X(\omega+\omega_0)\right]$$



C. Multi. of two signals

$x(t) \leftrightarrow X(\omega)$

$y(t) \leftrightarrow Y(\omega)$

$$\boxed{xy \leftrightarrow \frac{1}{2\pi} X(\omega) * Y(\omega)}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{+i\omega t}\, d\omega$$

$$X(\omega) = \cancel{\frac{1}{2\pi}} \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\, dt$$

$$x y e^{-j\omega t} dt$$

proof

$$\int_{-\infty}^{\infty} x\,y\,e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} e^{-j\omega t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\alpha)e^{j\alpha t}\,d\alpha\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\beta)e^{j\beta t}\right]dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\alpha)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\beta)\underbrace{\left\{\int_{-\infty}^{\infty} e^{-j(\omega-\alpha-\beta)t}\,dt\right\}}_{2\pi\delta(\omega-\alpha-\beta)}d\beta\right]d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\alpha)\left[Y(\beta)\,\delta(\omega-\alpha-\beta)\,d\beta\right]d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\alpha)\,Y(\omega-\alpha)\,d\alpha$$

$$\mathcal{F}(xy) = \frac{1}{2\pi}X(\omega) * Y(\omega)$$

Example

$$
\begin{tikzcd}
x(t) \arrow[r] & \otimes \arrow[r] & y(t) \\
& w(t) \arrow[u] &
\end{tikzcd}
$$

$$w(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

Find $Y(\omega)$

① $w(t)$ is periodic !! FS

$$w(t) = \sum_{n=-\infty}^{\infty} a_n e^{j\omega_0 n t}$$

$$a_n = \frac{1}{T_s} \int_0^{T_s} \delta(t) e^{-j\omega_0 n t} \, dt = \frac{1}{T_s}$$

$$\omega_0 = \frac{2\pi}{T_s}$$

$$\boxed{w(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T_s} e^{j\omega_0 n t}}$$

$$W(\omega) = \sum_{n=-\infty}^{\infty} \frac{1}{T_s} \int_{-\infty}^{\infty} e^{-j(\omega - n\omega_0)t} \, dt$$

$$\boxed{W(\omega) = \sum_{n=-\infty}^{\infty} \frac{1}{T_s} (2\pi) \delta(\omega - n\omega_0)}$$

(2)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

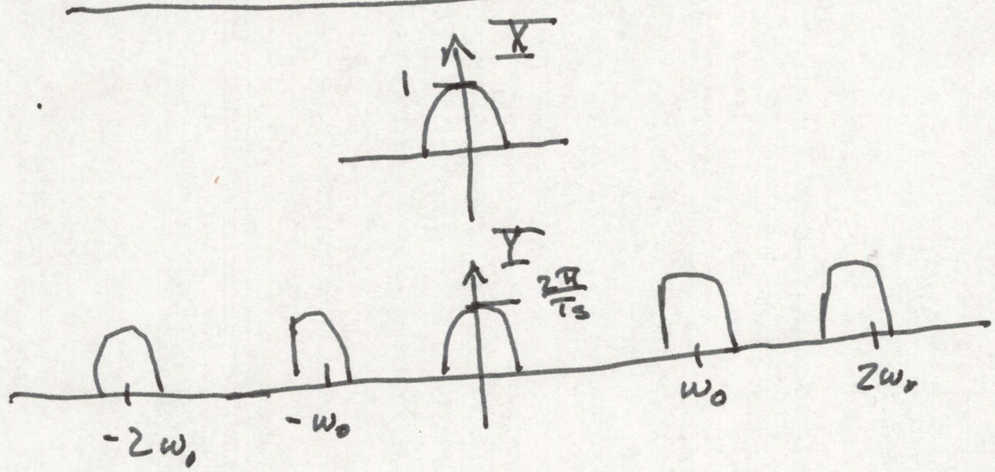
(3)

$$Y(\omega) = X(\omega) * W(\omega) \frac{1}{2\pi}$$

$$= X(\omega) * \left[\frac{1}{T_s} 2\pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) \right]$$

$$= \frac{2\pi}{T_s} \sum_{n=-\infty}^{\infty} X(\omega) * \delta(\omega - n\omega_0)$$

$$\boxed{Y(\omega) = \frac{2\pi}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_0)}$$



D.

$$x * y \longleftrightarrow X\,Y$$

proof

$$x * y = \int_{-\infty}^{\infty} x(\alpha)\, y(t-\alpha)\, d\alpha$$

$$\mathcal{F}(x*y) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\alpha)\, y(t-\alpha)\, d\alpha \right] e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(\alpha) \left[\int_{-\infty}^{\infty} y(t-\alpha)\, e^{-j\omega t}\, dt \right] d\alpha$$

$$t-\alpha = \beta \Rightarrow t = \alpha + \beta$$

$$dt = d\beta$$

$$= \int_{-\infty}^{\infty} x(\alpha) \underbrace{\left[\int_{-\infty}^{\infty} y(\beta)\, \cancel{e^{-j\omega\alpha}}\, e^{-j\omega\beta}\, d\beta \right]}_{Y(\omega)} e^{-j\omega\alpha}\, d\alpha$$

$$= \int_{-\infty}^{\infty} x(\alpha)\, e^{-j\omega\alpha}\, Y(\omega)\, d\alpha$$

$$\mathcal{F}(x*y) = X\,Y$$

Example

$$x(t) \longrightarrow \boxed{H(\omega)} \longrightarrow y(t)$$

given $$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{j\omega_0 n t}$$

$$H(\omega) = \frac{1}{j\omega + a}$$

1. Find $X(\omega)$

2. Find $Y(\omega)$

3. Find $y(t)$

soln

① $$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt =$$

$$= \int_{-\infty}^{\infty} \sum_n{}' a_n e^{-j(\omega - n\omega_0)t}\, dt$$

$$\boxed{X(\omega) = \sum_{n=-\infty}^{\infty}{}' a_n \, 2\pi \delta(\omega - n\omega_0)}$$

② $$Y(\omega) = H(\omega) X(\omega)$$

$$Y(\omega) = \sum_n{}' a_n 2\pi \left[\frac{\delta(\omega - n\omega_0)}{j\omega + a} \right]$$

③ $$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} 2\pi \sum_n{}' a_n \left[\int_{-\infty}^{\infty} \frac{\delta(\omega - n\omega_0) e^{j\omega t}}{j\omega + a}\, d\omega \right]$$

$$\boxed{y(t) = \sum_n{}' a_n \frac{e^{jn\omega_0 t}}{jn\omega_0 + a}}$$