Transient Analysis

SISO

Consider

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t)$$

$$G(s)$$

$$\boxed{Y(s) = G(s)\,X(s)} \qquad ROC_y = ROC_x \cap ROC_g$$

$$ROC_x : Re(s) > a$$

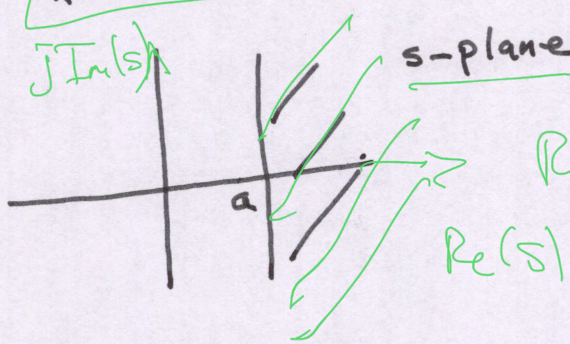
where

$$X(s) = \int_0^\infty x(t)\,e^{-st}\,dt$$

$$x(t) = \frac{1}{2\pi i}\int_{-i\infty + c}^{i\infty + c} X(s)\,e^{st}\,ds$$

$$\boxed{x(t) \longleftrightarrow X(s);\ ROC_x}$$

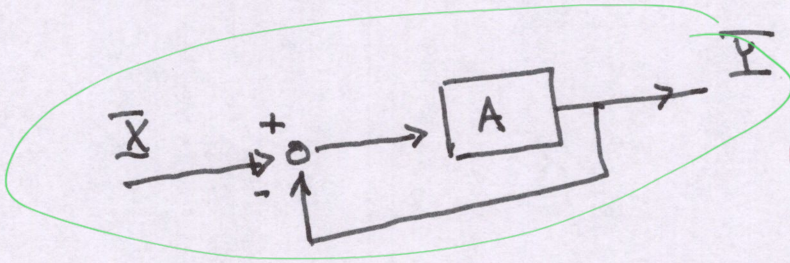
ROC_x :



$\therefore$

$Re(s) > a$

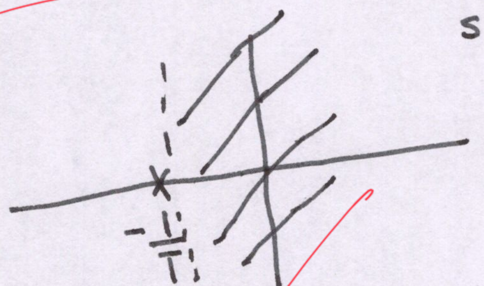
1^{st} order system



$$A = \frac{1}{Ts}$$

$$\boxed{\frac{Y}{X} = H(s) = \frac{A}{1+A}} = \boxed{\frac{1}{Ts+1} = H(s)}$$

s-plane



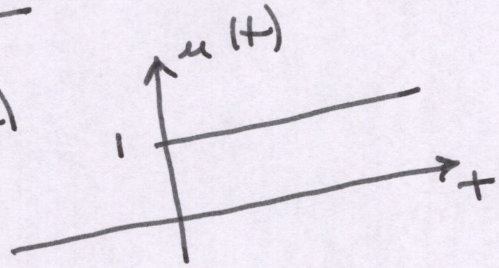
$$\boxed{ROC_h : Re(s) > -\frac{1}{T}}$$

system causality

$$u(t) = \begin{cases} 1 & t>0 \\ \frac{1}{2} & t=0 \\ 0 & t<0 \end{cases}$$

Step Response

$x(t) = u(t)$



$$X(s) = \int_0^\infty u(t)\, e^{-st}\, dt$$

$$= \left. \frac{e^{-st}}{-s} \right|_0^\infty$$

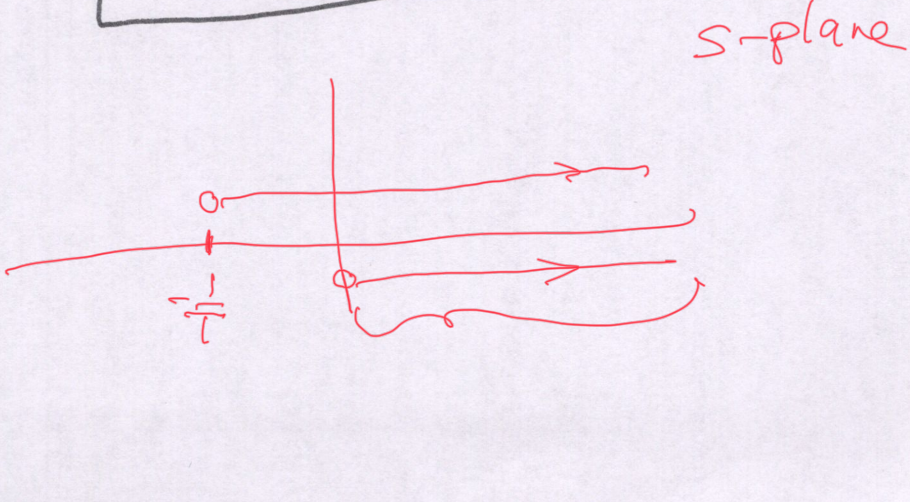
$$= \frac{-e^{-s0}}{-s} + \frac{e^{-s\infty}}{s} \implies \boxed{\begin{array}{c} \frac{1}{s} = X(s) \\ Re(s) > 0 \end{array}}$$

$$\boxed{Re(s) > 0}$$

$$\boxed{Y = \left[\frac{1}{Ts+1}\right] \frac{1}{s}}$$

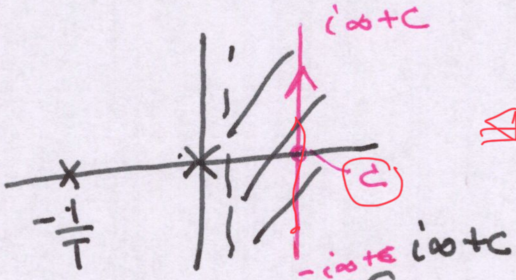
$$Re(s) > -\frac{1}{T} \cap Re(s) > 0$$

$$\boxed{ROC_y : Re(s) > 0}$$



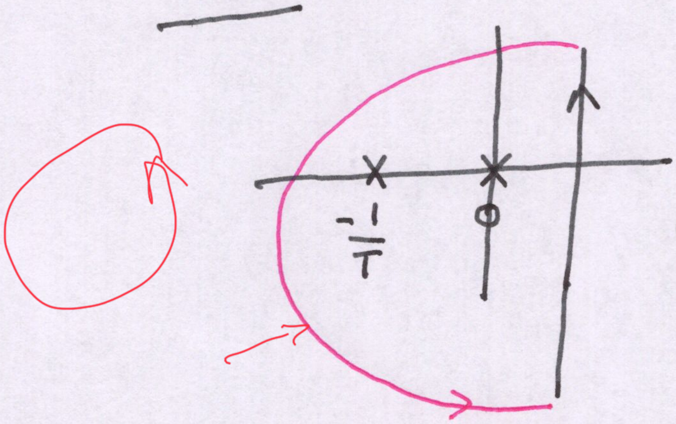
Find $y(t)$

$$\bar{Y}(s) = \frac{1}{s(Ts+1)} \qquad ROC_y \quad Re(s) > 0$$



$$y(t) = \frac{1}{2\pi i}\int_{-i\infty + c}^{i\infty + c} \bar{Y}(s)\, e^{st}\, ds$$

$t > 0$



$$\cancel{y(t) = \frac{1}{2\pi i}\int_{-i\infty + c}^{i\infty + c} \bar{Y}(s)\, e^{st}\, ds = \oint_{\mathbb{C}} \bar{Y}(s)\, e^{st}\, ds}$$

$$y(t) = \frac{1}{2\pi i}\int_{-i\infty + c}^{i\infty + c} \bar{Y} e^{st}\, ds = \frac{1}{2\pi i}\oint_{\mathbb{C}} \bar{Y} e^{st}\, ds$$

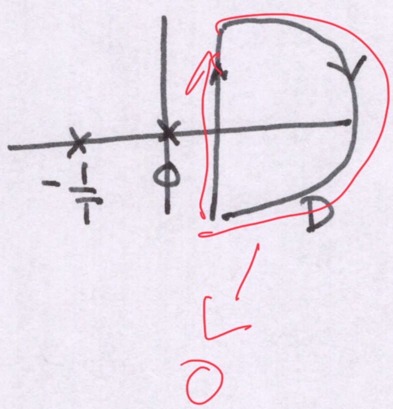
$t>0$

$$y(t) = \left[\frac{Y e^{st}}{2\pi i}\right] s\,(2\pi i)\Bigg|_{s=0} + \left[\frac{Y e^{st}}{2\pi i}\right]\left(s+\frac{1}{T}\right)(2\pi i)\Bigg|_{s=-\frac{1}{T}}$$

$$y(t) = \frac{e^{st}}{(Ts+1)}\Bigg|_{s=0} + \frac{e^{st}}{s}\frac{\left(s+\frac{1}{T}\right)}{(Ts+1)}\Bigg|_{s=-\frac{1}{T}}$$

$$\boxed{y(t) = 1 - e^{-t/T}} \quad t>0$$

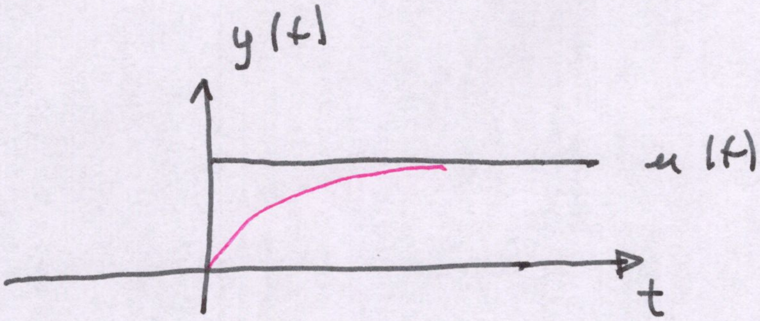
$t<0$



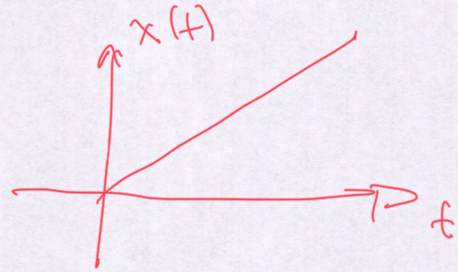
$$y(t) = \frac{1}{2\pi i}\oint_D Y(s)\, e^{st}\, ds$$

$$\boxed{y(t) = 0 \quad t<0}$$

via Cauchy - Gorsat thm



$$y(\infty) = \lim_{s \to 0} sY = \lim_{s \to 0} s\left[\frac{1}{s(\tau s+1)}\right] = 1 \checkmark$$



Ramp response

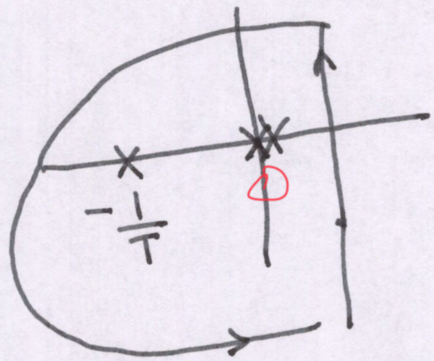
$$x(t) = t\,u(t)$$

$$X(s) = \frac{1}{s^2} \qquad ROC_x : Re(s) > 0$$

$$Y(s) = \frac{1}{s^2(Ts+1)} \qquad ROC_y : Re(s) > 0$$

$t > 0$

s-plane

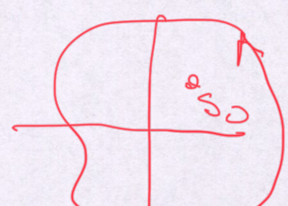


C

$$y(t) = \frac{1}{2\pi i} \oint_C Y(s) e^{st} ds$$

Cauchy Int. exten.

$$I = \oint \frac{A(s)}{(s - s_0)^n} ds$$



$$I = \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} A(s) \Big|_{s=s_0} (2\pi i)$$

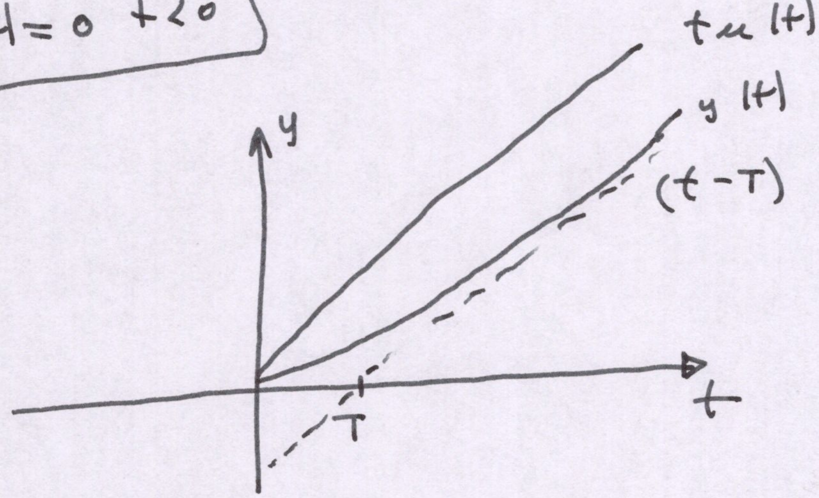
$$y(t) = \frac{1}{1!}\frac{d}{ds}\left[\frac{Y e^{st}}{2\pi i}\, s^2\right] 2\pi i \,\Bigg|_{s=0} + \left[\frac{Y e^{st}}{2\pi i}\right]\left(s+\frac{1}{T}\right) 2\pi i \,\Bigg|_{s=-\frac{1}{T}}$$

$$= \frac{d}{ds}\left[\frac{e^{st}}{Ts+1}\right]\Bigg|_{s=0} + \frac{e^{st}}{s^2}\frac{(s+1/T)}{(Ts+1)}\Bigg|_{s=-\frac{1}{T}}$$

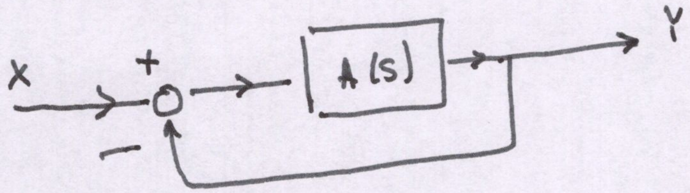
$$= \left[\frac{t e^{st}}{Ts+1} - \frac{T e^{st}}{(Ts+1)^2}\right]\Bigg|_{s=0} + \frac{e^{st}}{s^2 T}\Bigg|_{s=-\frac{1}{T}}$$

$$\boxed{y(t) = \left[t - T\right] + T e^{-t/T}} \quad \underline{t>0}$$

$$\boxed{y(t) = 0 \quad t<0}$$



Second Order



$$A(s) = \frac{\omega_n^2}{s(s+2\xi\omega_n)}$$

$$\frac{Y}{X} = \frac{A}{1+A} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

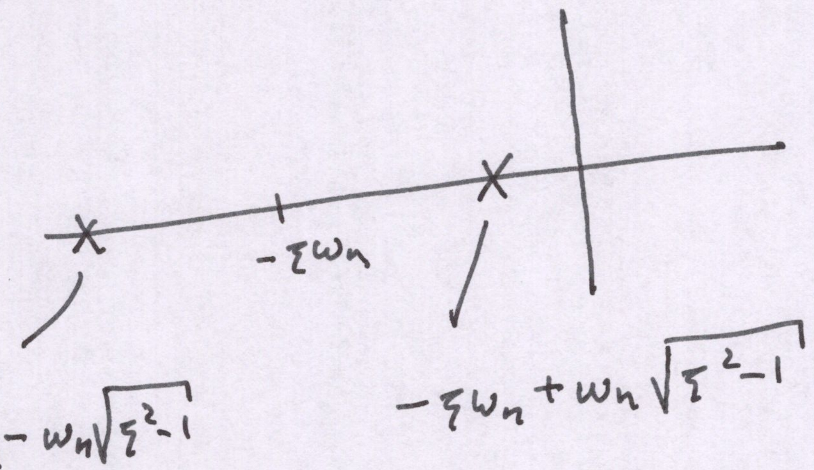
poles

$$s = -\xi\omega_n \pm \sqrt{\frac{(2\xi\omega_n)^2 - 4\omega_n^2}{2}}$$

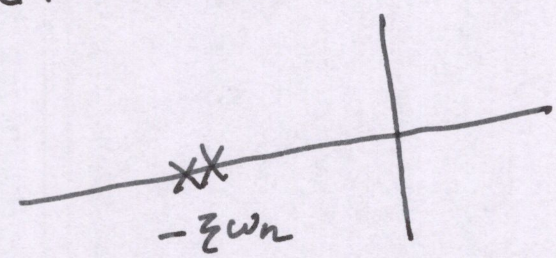
$$\boxed{s = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}}$$

$$S = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

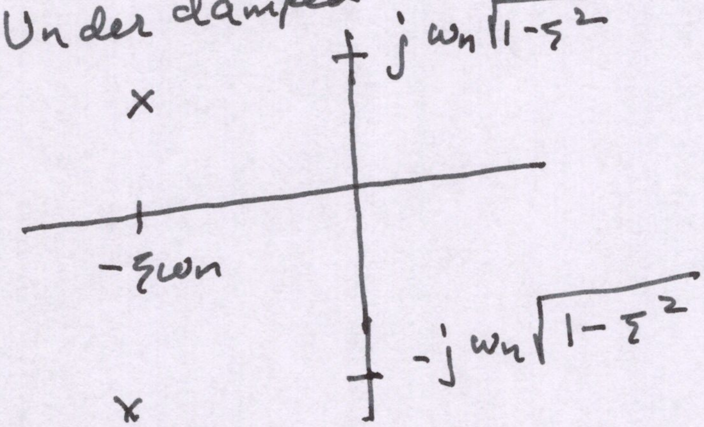
$\underline{\xi > 1}$ Overdamped



$\underline{\xi = 1}$ Critically damped



$1 > \xi > 0$ Under damped



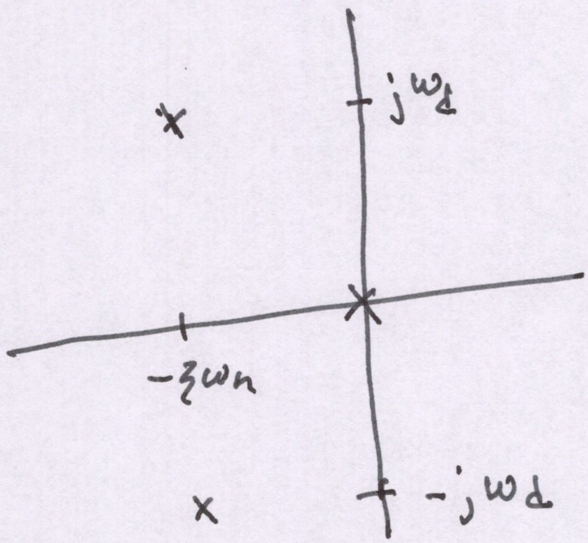
Step response of underdamped system

$$Y(s) = \left(\frac{1}{s}\right) \frac{\omega_n^2}{(s+\xi\omega_n + j\omega_d)(s+\xi\omega_n - j\omega_d)}$$

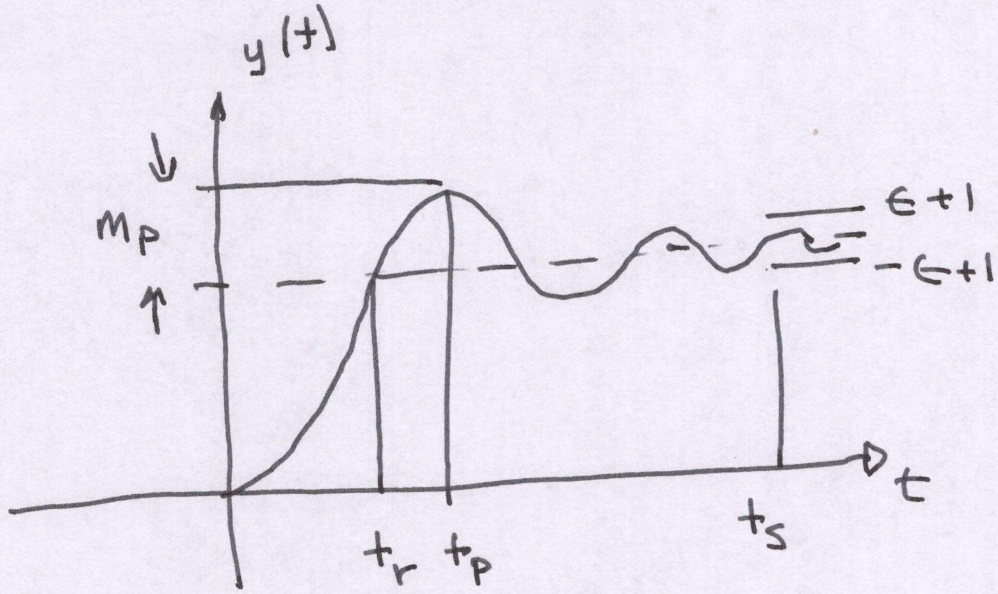
$$\omega_d = \omega_n\sqrt{1-\xi^2}$$ damped natural freq

natural freq

ω_n



$$y(t) = 1 - e^{-\xi\omega_n t}\left[\cos(\omega_d t) + \frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_d t)\right]$$


y(t)
Mp
∈+1
-∈+1
t
tr
tp
ts