Linear Quadratic Regulator (LQR)

problem statement

$$\min \quad PI = \int_0^\infty \underline{x}^T Q\, \underline{x} + \underline{u}^T P \underline{u} \; dt$$

subject to

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

Since $\int_{0}^{\infty}$ → infinite horizon ⇒ means $\boxed{\dot{R} = 0}$

The regulator satisfies
the algebric Ricatti eqn

$$RA + A^T R - R B P^{-1} B^T R + Q = 0$$

$$\underline{u}^0 = -\cancel{\tfrac{1}{2}} \left[P^{-1} B R \right] \underline{x}$$

example 1

$$\dot{x} = -x + u$$

$$PI = \int_0^\infty 2x^2 + \frac{u^2}{2}\,dt$$

① Find A, B, P, Q

$A = -1 \quad B = 1$

$P = \frac{1}{2} \quad Q = 2$

$R = r$

② ~~A.R.eqn~~ algebric Riccati eqn

$$r(-1) + (-1)^T r - r(1)\left(\tfrac{1}{2}\right)^{-1}(1)^T r + (2) = 0$$

$$-r \qquad -r \qquad -r(1)(2)(1)r + 2 = 0$$

$$\boxed{-2r^2 - 2r + 1 = 0}$$

Soln $r = \begin{cases} (-)\left[\frac{\sqrt{3}+1}{2}\right] \\ \frac{\sqrt{3}-1}{2} \end{cases} \Rightarrow$ R is pos def

$\therefore \boxed{r = \frac{\sqrt{3}-1}{2}}$

③ optimal input

$$\underline{u}^0 = -\frac{1}{2}\left[P^{-1}B^T R\right]\underline{x}$$

$$u^0 = -\frac{1}{2}\left[\left(\tfrac{1}{2}\right)^{-1}(1)\left(\frac{\sqrt{3}-1}{2}\right)\right]x$$

$$\boxed{u^0 = -\left(\frac{\sqrt{3}-1}{2}\right)x}$$

④ note $\dot{x} = -x + u^0 \Rightarrow \dot{x} = \left(-1 - \frac{\sqrt{3}}{2} + \frac{1}{2}\right)x \Rightarrow \dot{x} = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right)x$

eigen value
uncontrolled $(-1) \Rightarrow$ controlled $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right)$

example 2 Find u^o using HJ-approach

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$PI = \int_0^\infty (4x_1^2 + u^2)\,dt$$

1. Find A, B, Q, P $\quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad R = \begin{bmatrix} r_1 & r_2 \\ r_2 & r_3 \end{bmatrix}$

$P = 1 \quad Q = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \Leftarrow \underline{x}^T Q \underline{x}$

2. A.R.E.: $RA + A^T R - R B P^{-1} B^T R + Q = 0$

$$\underbrace{\begin{bmatrix} r_1 & r_2 \\ r_2 & r_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\begin{bmatrix} 0 & r_1 \\ 0 & r_2 \end{bmatrix}} + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} r_1 & r_2 \\ r_2 & r_3 \end{bmatrix}}_{\begin{bmatrix} 0 & 0 \\ r_1 & r_2 \end{bmatrix}} - \begin{bmatrix} r_1 & r_2 \\ r_2 & r_3 \end{bmatrix} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 \begin{bmatrix} 0 & 1 \end{bmatrix}}_{\underbrace{(1)\begin{bmatrix} 0\cdot 0 & 0\cdot 1 \\ 1\cdot 0 & 1\cdot 1 \end{bmatrix}}_{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}} \begin{bmatrix} r_1 & r_2 \\ r_2 & r_3 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 - r_2^2 & r_1 - r_2 r_3 \\ r_1 - r_2 r_3 & 2r_2 - r_3^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

matrix should be symetric!!!

$[A] \quad a_{ij} = a_{ji}$

3. Solve for R

$4 - r_2^2 = 0$

$\left.\begin{matrix} r_1 - r_2 r_3 = 0 \\ r_1 - r_2 r_3 = 0 \end{matrix}\right\}$ same ✓

$2r_2 - r_3^2 = 0$

$$4 - r_2^2 = 0 \Rightarrow r_2 = \pm 2 \Rightarrow \boxed{r_2 = 2}$$

$$2r_2 - r_3^2 = 0 \Rightarrow 4 - r_3^2 = 0 \Rightarrow r_3 = \pm 2 \Rightarrow \boxed{r_3 = 2}$$

$$r_1 - r_2 r_3 = 0 \Rightarrow r_1 - (2)(2) = 0 \Rightarrow \boxed{r_1 = 4}$$

$$R = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

4. optimal input

$$u^o = -P^{-1} B^T R \, \underline{x}$$

$$u^o = -(1)^{-1} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\boxed{u^o = -\begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2x_1 - 2x_2}$$

example 3   Determine the Alg. Ricatti eqn

$$\dot{\underline{x}} = \begin{bmatrix}0&1&0\\0&0&1\\0&-2&-3\end{bmatrix}\underline{x} + \begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}\underline{u}; \quad PI = \int_0^\infty x_1^2 + x_2^2 + 4u_1^2 + u_2^2 \, dt$$

① Find A, B, P, Q

$$A = \begin{bmatrix}0&1&0\\0&0&1\\0&-2&-3\end{bmatrix} \qquad B = \begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix} \qquad P = \begin{bmatrix}4&0\\0&1\end{bmatrix}$$

$$Q = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} \qquad P^{-1} = \frac{1}{4}\begin{bmatrix}1&0\\0&4\end{bmatrix}$$

$$R = \begin{bmatrix}r_{11}&r_{12}&r_{13}\\r_{21}&r_{22}&r_{23}\\r_{31}&r_{32}&r_{33}\end{bmatrix} \underset{\text{sym}}{\Rightarrow} \begin{bmatrix}r_{11}&r_{12}&r_{13}\\r_{12}&r_{22}&r_{23}\\r_{13}&r_{23}&r_{33}\end{bmatrix}$$

② $RA + A^TR - RBP^{-1}B^TR + Q = [0]$

$$R\begin{bmatrix}0&1&0\\0&0&1\\0&-2&3\end{bmatrix} + \begin{bmatrix}0&0&0\\1&0&-2\\0&1&-3\end{bmatrix}R - R\underbrace{\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}\left(\frac{1}{4}\right)\begin{bmatrix}1&0\\0&4\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}}R + \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} = \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}\left(\frac{1}{4}\right)\begin{bmatrix}1&0&0\\0&0&4\end{bmatrix}$$

$$\left(\frac{1}{4}\right)\begin{bmatrix}1&0&0\\0&0&0\\0&0&4\end{bmatrix}$$

$$\begin{bmatrix}1/4&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

example 4

$$\left.\begin{aligned}\dot{\underline{x}} &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ PI &= \int_0^\infty \alpha x_1^2 + u^2 \, dt \end{aligned}\right\} \text{given } \alpha = 1$$

$$Q = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}$$

1° $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $P = 1$

$R = \begin{bmatrix} r_1 & r_2 \\ r_2 & r_3 \end{bmatrix}$ $P^{-1} = \frac{1}{1}$

2° $$\begin{bmatrix} r_1 & r_2 \\ r_2 & r_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} r_1 & r_2 \\ r_2 & r_3 \end{bmatrix}$$

$$- \begin{bmatrix} r_1 & r_2 \\ r_2 & r_3 \end{bmatrix} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{1} \begin{bmatrix} 0 & 1 \end{bmatrix}}_{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} r_1 & r_2 \\ r_2 & r_3 \end{bmatrix} + \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -r_2^2 - 4r_2 + \alpha & -r_2 r_3 - 2r_3 - 3r_2 + r_1 \\ -r_2 r_3 - 2r_3 - 3r_2 + r_1 & -r_3^2 - 6r_3 + 2r_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3° Determine u^o for $\alpha = 1$

(a) $r_2^2 + 4r_2 = 1 \Rightarrow \boxed{r_2 = \sqrt{5} - 2}$

(b) $r_3^2 + 6r_3 = 2r_2 \Rightarrow r_3 = \sqrt{2\sqrt{5} + 5} - 3$

(c) $-r_2 r_3 - 2r_3 - 3r_2 + r_1 = 0 \Rightarrow r_1 = \sqrt{5}\sqrt{2\sqrt{5} + 5} - 6$

$$u^0 = -\left(\frac{1}{1}\right) \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 & r_2 \\ r_2 & r_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$