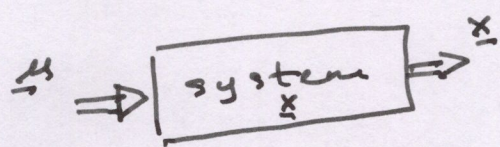


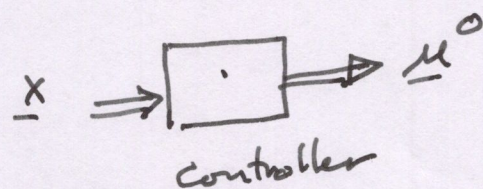
Optimal closed-loop control



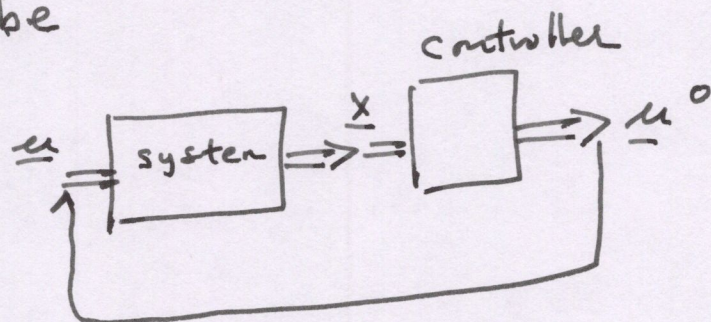
① we have knowledge of x "obs"

② any x is possible given the right u "cont"

Objective is to find the controller such that



The complete controlled system will be



~~In~~ In addition we will assume that u^0 is linear function of x

Problem statement

$$\min \quad PI = \int_t^{t_f} [\underline{x}^T Q \underline{x} + \underline{u}^T P \underline{u}] dt$$

subject to the constraint

$$\dot{\underline{x}} = A \underline{x} + B \underline{u}$$

note new features

① introduction of the control time horizon

$$PI = \int_t^{t_f} [\quad] dt$$

/ \

current time final time

ie inf horizons

$$PI = \int_0^{\infty} [\quad] dt$$

② Introduction quadratic form

$$\underline{x}^T Q \underline{x}$$

ii

$$\left. \begin{array}{l} \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{array} \right\} \left\{ \underline{x}^T Q \underline{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right.$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$$

$$= x_1(ax_1 + bx_2) + x_2(cx_1 + dx_2)$$

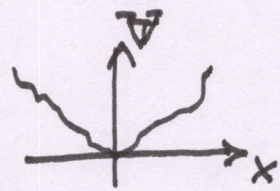
$$\underline{x}^T Q \underline{x}$$

$$\underline{x}^T Q \underline{x} = ax_1^2 + x_1x_2(b+c) + d \cdot x_2^2$$

Prop of $\underline{x}^T Q \underline{x} = V$

① V is positive definite

if $V > 0$ for all values of $\underline{x} \neq 0$

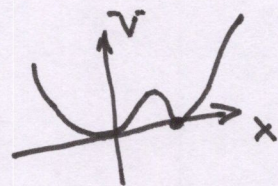


Therefore Q is symmetric and eigenvalues positive

① $Q_{ij} = Q_{ji}$

② $\lambda_i > 0$

② V is positive semidefinite
if $V \geq 0$ for all values of $\underline{x} \neq 0$



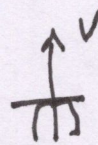
Therefore Q is symmetric and eigenvalues 0 or pos.

① $Q_{ij} = Q_{ji}$

② $\lambda_i \geq 0$

③ V negative definite

$Q_{ij} = Q_{ji} \quad \lambda_i < 0$



④ V negative semidef

$Q_{ij} = Q_{ji} \quad \lambda_i \leq 0$



⑤ If not ①-④ indefinite

Converting quadratic poly
to matrix form

$$V = x_1^2 + 0x_1x_2 + 6x_2^2$$

$$V = ax_1^2 + bx_1x_2 + cx_2^2$$

$$\begin{matrix} x_1 & x_2 \\ \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \end{matrix}$$

$$V = ax_1^2 + 6x_1x_3 + dx_2^2 + ex_3^2$$

$$\begin{matrix} x_1 & x_2 & x_3 \\ \begin{bmatrix} a & 0 & 6/2 \\ 0 & d & 0 \\ 6/2 & 0 & e \end{bmatrix} \end{matrix}$$

Hamilton - Jacobi Approach

$$PI = \int_t^{t_f} \underline{x}^T Q \underline{x} + \underline{u}^T P \underline{u} dt$$

$$\dot{\underline{x}} = A \underline{x} + B \underline{u}$$

assume

Q is positive-semidef

P is positive def

$$H = (\nabla V)^T [A \underline{x} + B \underline{u}] + \underline{x}^T Q \underline{x} + \underline{u}^T P \underline{u}$$

Some useful identities

$$\hat{V} = \underline{a}^T B \underline{c}$$

$$\frac{\partial \hat{V}}{\partial \underline{a}} = (\underline{B} \underline{c})^T = \underline{c}^T B^T \cdot \underline{B} \underline{c}$$

$$\frac{\partial \hat{V}}{\partial \underline{c}} = (\underline{a}^T B)^T = B^T (\underline{a}^T)^T = B^T \underline{a}$$

Find the optimal input

$$\frac{\partial H}{\partial \underline{u}} = 0 = [(\nabla V)^T B]^T + P \underline{u} + (\underline{u}^T P)^T$$

$$0 = B^T (\nabla V) + P \underline{u} + P^T \underline{u}$$

not P is sym. $P = P^T$

$$0 = B^T (\nabla V) + 2P \underline{u}$$

Solve for $\underline{u}$

$$\underline{u}^0 = -\frac{1}{2} P^{-1} [B^T \nabla V]$$

Final H^0

$$H^0 = H \Big|_{\underline{u} = \underline{u}^0}$$

$$\begin{aligned}
 H^0 &= (\nabla V)^T A \underline{x} + (\nabla V)^T B \left[P^{-1} B^T (\nabla V) \right] \frac{1}{2} (-1) \\
 &+ \underline{x}^T Q \underline{x} \\
 &+ \frac{1}{4} \left[P^{-1} B^T (\nabla V) \right]^T P \left[P^{-1} B^T (\nabla V) \right]
 \end{aligned}$$

side calc.

$$\begin{aligned}
 \frac{1}{4} \left[P^{-1} B^T (\nabla V) \right]^T &= \frac{1}{4} \left[(B^T \nabla V)^T (P^{-1})^T \right] \\
 &= \frac{1}{4} \left[(\nabla V)^T B (P^{-1})^T \right] \\
 &= \frac{1}{4} \left[(\nabla V)^T B P^{-1} \right]
 \end{aligned}$$

note
 $(P^{-1})^T = P^{-1}$

$$\begin{aligned}
 H^0 &= (\nabla V)^T A \underline{x} - \frac{1}{2} (\nabla V)^T B P^{-1} B^T (\nabla V) + \underline{x}^T Q \underline{x} \\
 &+ \frac{1}{4} (\nabla V)^T B P^{-1} \underbrace{P P^{-1}}_I B^T (\nabla V)
 \end{aligned}$$

$$H^0 = (\nabla V)^T A \underline{x} - \frac{1}{4} (\nabla V)^T B P^{-1} B^T (\nabla V) + \underline{x}^T Q \underline{x}$$

given HJ - equation

$$\textcircled{1} \quad \frac{\partial V}{\partial t} + H^0 = 0$$

$$\textcircled{2} \quad V = \underline{x}^T R \underline{x} \quad R \text{ is positive def} \quad R = R^T$$

then

$$\left. \frac{\partial V}{\partial t} \right|_{\text{holding } \underline{x} \text{ cons}} = \underline{x}^T \dot{R} \underline{x}$$

$$(\nabla V) = R \underline{x} + R^T \underline{x} = (R + R^T) \underline{x} = 2R \underline{x}$$

$$\frac{\partial V}{\partial t} + H^0 = 0$$

$$\underline{x}^T \dot{R} \underline{x} + 2 \underline{x}^T R A \underline{x} - \underline{x}^T R B P^{-1} B^T R \underline{x} + \underline{x}^T Q \underline{x} = 0$$

$$\underline{x}^T \left[\dot{R} + 2RA - RB P^{-1} B^T R + Q \right] \underline{x} = 0$$

$$\therefore \boxed{\dot{R} + 2RA - RB P^{-1} B^T R + Q = 0}$$

Since R is to remain sym for all time take the sym. part of

$$(2RA)$$

$$(2RA)_{\text{sym}} = (RA) + (RA)^T$$

$$(2RA)_{\text{sym}} = RA + A^T R$$

$$\dot{R} + RA + A^T R - R B P^{-1} B^T R + Q = 0$$

$$\text{where } R|_{t=t_f} = [0]$$

The optimal input

$$\underline{u}^0 = -\frac{1}{2} P^{-1} B (\nabla V) \quad \leftarrow \nabla V = 2 R \underline{x}$$

$$\underline{u}_0^0 = -\frac{1}{2} [P^{-1} B R] \underline{x}(2) \leftarrow$$

