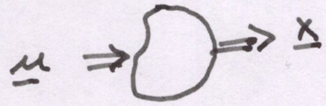
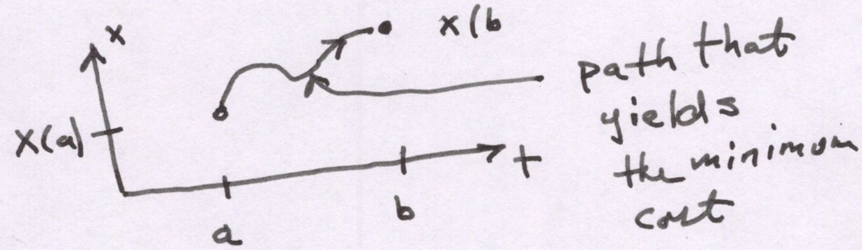
Optimal Open-Loop control



objective: Find the optimal input $\underline{u}^o$ such that the "cost" over a time interval $t=[a,b]$ is minimized.

example



Math statement

Minimize

$$PI = \int_a^b L(\underline{x}, \underline{u})\,dt$$

"cost" or performance index

Subject to the constraint

$$\underline{0} = \underline{f}(\underline{x}, \underline{u}, t) - \dot{\underline{x}}$$

Solution Approach : Method of Lagrange Multiplier (1760)

Single statement of the problem:

minimize $$V = \int_a^b \left[\mathcal{L} + \underline{\lambda}^T \left\{ \underline{f} - \underline{\dot{x}} \right\} \right] dt$$

integration by parts

$$\int_a^b \underline{\lambda}^T \underline{\dot{x}} = \underline{\lambda}^T \underline{x} \Big|_a^b - \int_a^b \underline{\dot{\lambda}}^T \underline{x} \, dt$$

$$\boxed{V = \int_a^b \left\{ \left[\mathcal{L} + \underline{\lambda}^T \underline{f} \right] + \underline{\dot{\lambda}}^T \underline{x} \right\} dt - \underline{\lambda}^T \underline{x} \Big|_a^b}$$

where $\underline{\lambda}$ is the Lagrange multiplier vector

unknowns: $\underline{u}$ input
$\underline{x}$ state vector
$\underline{\lambda}$ Lagrange multiplier

→ To find the unknowns we must minimize V

example

Find V given

$$PI = \int_a^b \left[x_1^2 + 2x_2^2 + u^2\right] dt$$

constraint

$$0 = u + ax_1 + bx_2 - \dot{x}_1$$
$$0 = 6u + cx_1 + dx_2 - \dot{x}_2$$

$$\mathcal{L} = x_1^2 + 2x_2^2 + u^2 \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \underline{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \quad \underline{\lambda}^T = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix}$$

$$\underline{f} = \begin{bmatrix} u + ax_1 + bx_2 \\ 6u + cx_1 + dx_2 \end{bmatrix}$$

$$V = \int_a^b \left[\mathcal{L} + \underline{\lambda}^T \underline{f} + \dot{\underline{\lambda}}^T \underline{x}\right] dt - \underline{\lambda}^T \underline{x} \Big|_a^b$$

$$V = \int_a^b \left[x_1^2 + 2x_2^2 + u^2 + \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} u + ax_1 + bx_2 \\ 6u + cx_1 + dx_2 \end{bmatrix} + \begin{bmatrix} \dot{\lambda}_1 & \dot{\lambda}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right] dt$$

$$- \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Bigg|_a^b$$

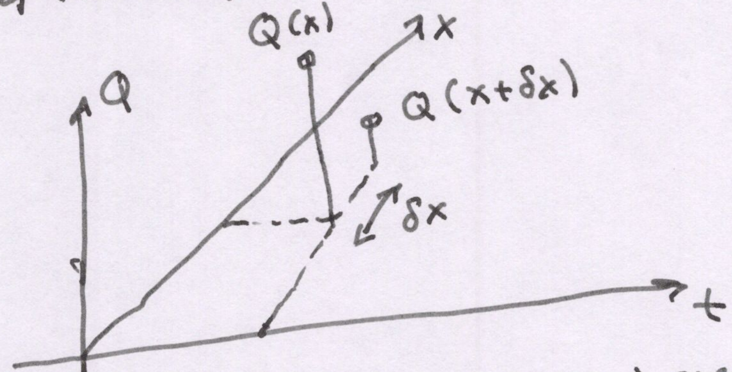
$$V = \int_a^b \left\{ x_1^2 + 2x_2^2 + u^2 + \lambda_1(u + ax_1 + bx_2) + \lambda_2(6u + cx_1 + dx_2) + \dot{\lambda}_1 x_1 + \dot{\lambda}_2 x_2 \right\} dt$$

$$- \left\{\lambda_1 x_1 + \lambda_2 x_2\right\} \Big|_a^b$$

Minimizing V Calculus of Variation

Consider the functional $Q(x)$ where ~~$x(t)$~~ x is a function of t

x is the dependent variable
t is the independent variable



To find min take the first increment of $Q(x)$ and set result equal to zero

$$\delta Q(x) = \left.\frac{\partial Q}{\partial x}\right|_{t\,\text{fixed}} \delta x = 0$$

implied x is fixed

soln

1. $\delta x = 0$
2. $\dfrac{\partial Q}{\partial x} = 0$

example

$$Q(\underline{x}) \quad \text{where} \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find δQ

$$\delta Q = \frac{\partial Q}{\partial x_1}\delta x_1 + \frac{\partial Q}{\partial x_2}\delta x_2 = \sum_{i=1}^{2} \frac{\partial Q}{\partial x_i}\delta x_i$$

$$\boxed{\delta Q = \left(\frac{\partial Q}{\partial \underline{x}}\right)^T \delta \underline{x}}$$

where $\dfrac{\partial Q}{\partial \underline{x}} = \begin{bmatrix} \dfrac{\partial Q}{\partial x_1} \\ \dfrac{\partial Q}{\partial x_2} \end{bmatrix}$ $\quad \delta \underline{x} = \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$

example

$$\cancel{\delta\left(\frac{dQ}{dt}\right) = \frac{d}{dt}\delta Q =}$$

Back to Discussion

$$V = \int_a^b \left\{ \underbrace{\left[\mathcal{L} + \underline{\lambda}^T \underline{f} \right]}_{H} + \dot{\underline{\lambda}}^T \underline{x} \right\} dt - \underline{\lambda}^T \underline{x} \Big|_a^b$$

$$V = \int_a^b \left\{ H + \dot{\underline{\lambda}}^T \underline{x} \right\} dt - \underline{\lambda}^T \underline{x} \Big|_a^b$$

where $H = \mathcal{L}(\underline{x}, \underline{u})$ State function of Pontrygin

$\min V$

$$\delta V = \int_a^b \delta \left\{ H + \dot{\underline{\lambda}}^T \underline{x} \right\} dt - \delta\left(\underline{\lambda}^T \underline{x} \right) \Big|_a^b = 0$$

~~$\delta V =$ $\delta V = 0$ invariant on a limit~~

$$\delta \left\{ H + \dot{\underline{\lambda}}^T \underline{x} \right\} = 0 \Rightarrow \left[\frac{\partial H}{\partial \underline{x}} + \dot{\underline{\lambda}} \right]^T \delta \underline{x} + \left[\frac{\partial}{\partial \underline{u}} H \right] \delta \underline{u} = 0$$

$$\delta \left(\underline{\lambda}^T \underline{x} \right) = 0 \Rightarrow \underline{\lambda}^T \delta \underline{x} \Big|_a = 0$$

$$\underline{\lambda}^T \delta \underline{x} \Big|_b = 0$$

$$-\dot{\underline{\lambda}} = \frac{\partial H}{\partial \underline{x}}$$

$$\frac{\partial H}{\partial \underline{u}} = 0$$

$$\dot{\underline{x}} = \underline{f}$$

with BC

$$\underline{\lambda}^T \delta \underline{x} \big|_a = 0$$

$$\underline{\lambda}^T \delta \underline{x} \big|_b = 0$$

and $H = \mathcal{L} + \underline{\lambda}^T f$

Example Find optimal input u

~~given~~ that min.

$$PI = \int_0^1 u^2\,dt$$

subject to the constraint

$$\dot{x} = x + u$$

BC $x(0) = 0$
$x(1) = 1$

(1) Find SFP

$$H = u^2 + \lambda(x+u)$$

(2) optimal u

$$\frac{\partial H}{\partial u} = 0 = 2u + \lambda \Rightarrow \boxed{u^\circ = -\frac{\lambda}{2}}$$

$$\boxed{H^\circ = \frac{\lambda^2}{4} + \lambda(x - \lambda/2)} = H\Big|_{u=-\lambda/2}$$

(3) coenergy eqn

$$-\dot{\lambda} = \frac{\partial H^\circ}{\partial x} \Rightarrow \boxed{-\dot{\lambda} = \lambda}$$

$$\boxed{\dot{x} = -\frac{\lambda}{2}}$$

(4)

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

(5) <u>Find STM</u>

eigenvalue/vector $\quad \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \beta_1 = 1$

$\underline{e}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \beta_2 \; -1$

$$\left.\begin{aligned} E &= \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix} \\ E^{-1} &= \begin{bmatrix} 1 & -1/4 \\ 1 & 1/4 \end{bmatrix} \\ e^{\Lambda t} &= \begin{bmatrix} e^{t} & \\ & e^{-t} \end{bmatrix} \end{aligned}\right\} \quad \Phi = E e^{\Lambda t} E^{-1} = \begin{bmatrix} e^{t} & \frac{e^{-t} - e^{t}}{4} \\ 0 & e^{-t} \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} e^{t} & \frac{e^{-t} - e^{t}}{4} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}$$

(6) given $x(0) = 0$

$$x(1) = 1 = \left(e^{t} x(0) + \lambda(0) \left[\frac{e^{-t} - e^{t}}{4} \right] \right) \Big|_{t=1}$$

$$\Rightarrow \lambda(0) = \left[\frac{e^{-1} - e^{1}}{4} \right]^{-1}$$

(7)

$$\boxed{\begin{aligned} \lambda(t) &= e^{-t} \left(\frac{4}{e^{-1} - e^{1}} \right) \\ u^{0} &= -\frac{1}{2} \lambda(t) \end{aligned}}$$

Setup SFP

problem $\quad \min PI = \int_a^b \mathcal{L}(\underline{x}, \underline{u})\,dt$

subject to $\quad \dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u})$

SFP

① $H = \mathcal{L} + \underline{\lambda}^T \underline{f}$

Find $\underline{u}$ such that H ~~is~~ has minimum value

~~assume $\frac{\partial H}{\partial \underline{u}}$ can be diff~~

② $\frac{\partial H}{\partial \underline{u}} = 0 \Rightarrow \underline{u}^{\circ}$

③ $H^{\circ} = H \Big|_{\underline{u} = \underline{u}^{\circ}}$

④ $\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}^{\circ})$

$-\dot{\underline{\lambda}} = \frac{\partial H^{\circ}}{\partial \underline{x}}$

$$\boxed{\text{problem} \rightarrow \text{ODEs}}$$

ex 1:

$$\dot{x}_1 = x_1 + x_2 + u$$
$$\dot{x}_2 = -2x_1$$
$$PI = \int_0^1 (u^2 + x_1^2)\,dt$$

$$H = (u^2 + x_1^2) + \lambda_1(x_1 + x_2 + u) + \lambda_2(-2x_1)$$

$$\frac{\partial H}{\partial u} = 0 = 2u + \lambda_1 \implies u^o = -\frac{\lambda_1}{2}$$

$$H^o = \left(-\frac{\lambda_1}{2}\right)^2 + x_1^2 + \lambda_1\left(x_1 + x_2 - \frac{\lambda_1}{2}\right) + (-2x_1)\lambda_2$$

$$\dot{x}_1 = x_1 + x_2 - \frac{\lambda_1}{2}$$
$$\dot{x}_2 = -2x_1$$
$$-\dot{\lambda}_1 = \frac{\partial H^o}{\partial x_1} = 2x_1 + \lambda_1 - 2\lambda_2$$
$$-\dot{\lambda}_2 = \frac{\partial H^o}{\partial x_2} = \lambda_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & \lambda_1 & \lambda_2 \end{array} \\ \begin{bmatrix} 1 & 1 & -1/2 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & -1 & 2 \\ 0 & 0 & -1 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix}$$

ex 2:

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$PI = \int_0^1 \left[x_1^2 + \frac{1}{2} x_2^2 + 6u^2 \right] dt$$

$$H = \left(x_1^2 + \frac{1}{2} x_2^2 + 6u^2 \right) + \lambda_1 x_2 + \lambda_2 \left(-2x_1 - 3x_2 + u \right)$$

$$\frac{\partial H}{\partial u} = 0 = 6 \cdot 2u + \lambda_2 = 0 \implies u^o = -\frac{\lambda_2}{12}$$

$$H^o = \left[x_1^2 + \frac{1}{2} x_2^2 + 6\left(\frac{-\lambda_2}{12}\right)^2 \right] + \lambda_1 x_2 + \lambda_2 \left[-2x_1 - 3x_2 - \frac{\lambda_2}{12} \right]$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2x_1 - 3x_2 - \frac{\lambda_2}{12}$$

$$\dot{\lambda}_1 = -\frac{\partial H^o}{\partial x_1} = \left(2x_1 - 2\lambda_2 \right)(-1)$$

$$\dot{\lambda}_2 = -\frac{\partial H^o}{\partial x_2} = \left(x_2 + \lambda_1 - 3\lambda_2 \right)(-1)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & -\frac{1}{12} \\ -2 & 0 & 0 & 2 \\ 0 & -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix}$$

ex3

$$\dot{\underline{x}} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \underline{x} + \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \underline{u}$$

$$PI = \int_a^b (x_1^2 + u_1^2 + u_2^2)\,dt$$

$$H = (x_1^2 + u_1^2 + u_2^2) + \lambda_1 (x_1 + x_2 + \alpha u_1) + \lambda_2 [-x_2 + \beta u_2]$$

$$\frac{\partial H}{\partial u_1} = 2u_1 + \alpha \lambda_1 = 0 \implies u_1 = -\frac{\alpha}{2}\lambda_1$$

$$\frac{\partial H}{\partial u_2} = 2u_2 + \beta \lambda_2 = 0 \implies u_2 = -\beta \frac{\lambda_2}{2}$$

$$H^0 = \left[x_1^2 + \left(-\frac{\alpha}{2}\lambda_1\right)^2 + \left(-\frac{\beta}{2}\lambda_2\right)^2\right] + \lambda_1 \left[x_1 + x_2 + \alpha\left(-\frac{\alpha}{2}\lambda_1\right)\right]$$
$$+ \lambda_2 \left[-x_2 + \beta\left(-\beta \frac{\lambda_2}{2}\right)\right]$$

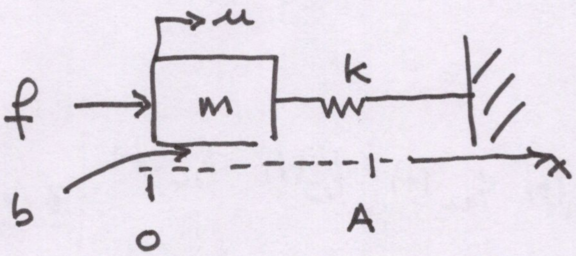
$$\dot{x}_1 = x_1 + x_2 + \alpha\left(-\frac{\alpha}{2}\lambda_1\right)$$

$$\dot{x}_2 = -x_2 + \beta\left(-\beta\frac{\lambda_2}{2}\right)$$

$$\dot{\lambda}_1 = -\frac{\partial H^0}{\partial x_1} = [2x_1 + \lambda_1](-1)$$

$$\dot{\lambda}_2 = -\frac{\partial H^0}{\partial x_2} = [\lambda_1 - \lambda_2](-1)$$

example



Find the input $\frac{f}{m}$ that minimized the effort energy

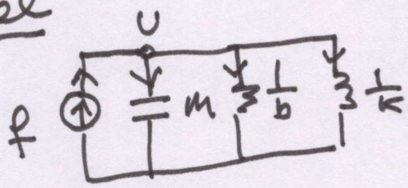
$$PI = \int_0^1 \hat{u}^2 dt$$

where $\hat{u} \equiv \frac{f}{m}$

in moving the mass from $x\big|_{t=0} = 0$ to $x\big|_{t=1} = A$

at $t=0$ $\dot{x}=0$ & $t=1$ $\dot{x}=0$

model



$$f = M\dot{u} + bu + k\int u\,dt$$

u (velocity) $= \dot{x}$

$$\boxed{f = M\ddot{x} + b\dot{x} + kx} \longrightarrow \left\{ \begin{array}{l} x_1 = x \\ x_2 = \dot{x} \\ f = M\dot{x}_2 + bx_2 + kx_1 \\ \dot{x}_1 = x_2 \end{array} \right.$$

state equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{f}{m}$$

rewritten

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 \\ -\alpha & -\beta \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u} \quad \text{where} \quad \alpha = \frac{k}{M} \quad \hat{u} = \frac{f}{M}$$

$$\beta = \frac{b}{M}$$

Design condition

$$\min \quad PI = \int_0^1 \hat{u}^2 \, dt$$

Boundary condition

$$\underline{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underline{x}(1) = \begin{bmatrix} A \\ 0 \end{bmatrix}.$$

Find optimal input

SFP

$$H = \hat{u}^2 + \lambda_1 x_2 + \lambda_2 \left[-\alpha x_1 - \beta x_2 + \hat{u} \right]$$

~~outp~~

optimal input

$$\frac{\partial H}{\partial \hat{u}} = 0 = 2\hat{u} + \lambda_2 = 0 \qquad \boxed{\hat{u}^0 = -\frac{\lambda_2}{2}}$$

H^0

$$H^0 = \left(-\frac{\lambda_2}{2} \right)^2 + \lambda_1 x_2 + \lambda_2 \left[-\alpha x_1 - \beta x_2 - \frac{\lambda_2}{2} \right]$$

governing eqns

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\alpha x_1 - \beta x_2 - \frac{\lambda_2}{2}$$

$$\dot{\lambda}_1 = -\frac{\partial H^0}{\partial x_1} = \alpha \lambda_2$$

$$\dot{\lambda}_2 = -\frac{\partial H^0}{\partial x_2} = -\lambda_1 + \beta \lambda_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\alpha & -\beta & 0 & -\lambda_2/2 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -1 & \beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix}$$

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soln

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = \left[\Phi(t) \right] \begin{bmatrix} \cancelto{0}{x_1(0)} \\ \cancelto{0}{x_2(0)} \\ \lambda_1(0) \\ \lambda_2(0) \end{bmatrix} \begin{matrix} \text{given} \\ \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{unknown} \end{matrix}$$

1

at $t = 1$

A

given

0

$$\begin{bmatrix} x_1(1) \\ x_2(1) \\ \lambda_1(1) \\ \lambda_2(1) \end{bmatrix} = \begin{bmatrix} \Phi(1) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \lambda_1(0) \\ \lambda_2(0) \end{bmatrix}$$

Find $\lambda_1(0)$ & $\lambda_2(0)$ first 2 eqns

$$A = \cancel{\phi_{11}(1)0} + \cancel{\phi_{12}(1)0} + \phi_{13}(1)\lambda_1(0) + \phi_{14}(1)\lambda_2(0)$$

$$0 = \cancel{\phi_{21}(1)0} + \cancel{\phi_{22}(1)0} + \phi_{23}(1)\lambda_1(0) + \phi_{24}(1)\lambda_2(0)$$

$$\begin{bmatrix} A \\ 0 \end{bmatrix} = \begin{bmatrix} \phi_{13}(1) & \phi_{14}(1) \\ \phi_{23}(1) & \phi_{24}(1) \end{bmatrix} \begin{bmatrix} \lambda_1(0) \\ \lambda_2(0) \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \lambda_1(0) \\ \lambda_2(0) \end{bmatrix} = \begin{bmatrix} \phi_{13}(1) & \phi_{14}(1) \\ \phi_{23}(1) & \phi_{24}(1) \end{bmatrix}^{-1} \begin{bmatrix} A \\ 0 \end{bmatrix}}$$

optimal input

$$\hat{u}^{o} = -\frac{\lambda_2}{2}$$

<u>using the STM</u>

$$\lambda_2(t) = \cancel{\phi_{41}(t)\, x_1(0)}^{\,0} + \cancel{\phi_{42}(t)\, x_2(0)}^{\,0} + \phi_{43}(t)\,\lambda_1(0) + \phi_{44}(t)\,\lambda_2(0)$$

$$\boxed{\hat{u}^{o} = -\frac{1}{2}\left[\phi_{43}(t)\,\lambda_1(0) + \phi_{44}(t)\,\lambda_2(0)\right]}$$