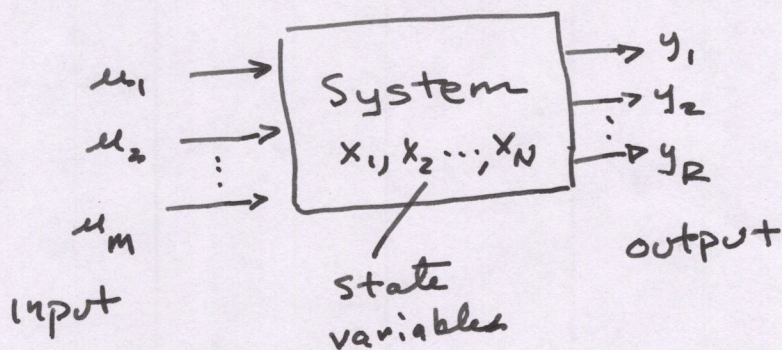


Modern Control

State Equations



input vector $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$

output vector $\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_R \end{bmatrix}$

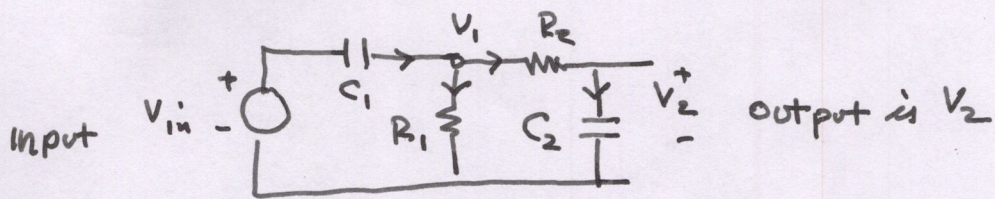
state vector $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$

Any LTI system can be expressed in matrix form as

$$\dot{\underline{x}} = [\underline{A}]\underline{x} + [\underline{B}]\underline{u}$$

$$\underline{y} = [\underline{C}]\underline{x} + [\underline{D}]\underline{u} \rightarrow \text{by assumption in this course}$$

Consider



using KCL

$$\frac{dv_1}{dt} = v_1 \left\{ -\frac{1}{R_1 C_1} - \frac{1}{R_2 C_1} \right\} + v_2 \left\{ \frac{1}{C_1 R_2} \right\} + \frac{dv_{in}}{dt}$$

$$\frac{dv_2}{dt} = v_1 \left\{ +\frac{1}{C_2 R_2} \right\} + v_2 \left\{ -\frac{1}{C_2 R_2} \right\}$$

State variable $v_1, v_2 \Rightarrow x_1 \equiv v_1$
 $x_2 \equiv v_2$

input $\frac{dv_{in}}{dt} \Rightarrow u_1$

output $v_2 \Rightarrow y = V_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1 C_1} - \frac{1}{R_2 C_1} & \frac{1}{C_1 R_2} \\ \frac{1}{C_2 R_2} & -\frac{1}{C_2 R_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Proof by example that $\underline{z}(t) = e^{\Lambda t} \underline{z}(0)$

consider

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\dot{z}_1 = \lambda_1 z_1 \Rightarrow z_1(t) = e^{\lambda_1 t} z_1(0)$$

$$\dot{z}_2 = \lambda_2 z_2 \Rightarrow z_2(t) = e^{\lambda_2 t} z_2(0)$$

in matrix form

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix}$$

Step 4 Transform back to $\underline{x}(t)$

$$\underline{x}(t) = E \underline{z}(t)$$

$\Downarrow$

$$E^{-1} \underline{x}(t) = \underline{z}(t)$$

then $\underline{z}(t) = e^{\Lambda t} \underline{z}(0)$

$$\Downarrow$$
$$E^{-1} \underline{x}(t) = e^{\Lambda t} E^{-1} \underline{x}(0)$$

$\Downarrow$

$$\underline{x}(t) = \left(E e^{\Lambda t} E^{-1} \right) \underline{x}(0)$$

Back to the general problem

if A has unique eigenvalues

$$\lambda\text{'s} = \lambda_1, \lambda_2, \lambda_3 \dots \lambda_N \quad \lambda_i \neq \lambda_j \text{ when } i \neq j$$

$$\underline{e}\text{'s} \Rightarrow \underline{e}_1, \underline{e}_2 \dots \underline{e}_N$$

Then

$$A \underline{e}_1 = \lambda_1 \underline{e}_1$$

$$A \underline{e}_2 = \lambda_2 \underline{e}_2$$

$$\vdots$$

$$A \underline{e}_N = \lambda_N \underline{e}_N$$

$$\left. \begin{array}{l} A \underline{e}_1 = \lambda_1 \underline{e}_1 \\ A \underline{e}_2 = \lambda_2 \underline{e}_2 \\ \vdots \\ A \underline{e}_N = \lambda_N \underline{e}_N \end{array} \right\} A [\underline{e}_1 | \underline{e}_2 \dots | \underline{e}_N] = [\lambda_1 \underline{e}_1 | \lambda_2 \underline{e}_2 \dots \lambda_N \underline{e}_N]$$

$$\underbrace{A}_{A} \underbrace{[\underline{e}_1 | \underline{e}_2 \dots \underline{e}_N]}_E = \underbrace{[\underline{e}_1 | \underline{e}_2 \dots \underline{e}_N]}_E \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix}}_{\Lambda}$$

$$AE = E\Lambda$$

$$\boxed{A = E\Lambda E^{-1}}$$

where E is comprised of the eigenvectors of A

Summary of the result

$$\underline{c}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \lambda_1 = 2$$

$$\underline{c}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \lambda_2 = 1$$

The solution

$$\underline{x} = B_1 \underline{c}_1 e^{\lambda_1 t} + B_2 \underline{c}_2 e^{\lambda_2 t}$$

$$\therefore \begin{cases} x_1 = B_1 (0) e^{2t} + B_2 (1) e^{1t} \\ x_2 = B_1 (1) e^{2t} + B_2 (-1) e^{1t} \end{cases}$$

Once we have the matrices A, B, C
how do we solve for $\underline{x}$?

Homogeneous soln

$$\text{given } \dot{\underline{x}} = A\underline{x} + B\underline{u}$$

$$\text{let } \underline{u} = 0$$

$$\dot{\underline{x}} = A\underline{x}$$

$$\text{say } \underline{x} = e^{\lambda t} \underline{c} \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_N \end{bmatrix} = e^{\lambda t} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}$$

$$\text{then } \dot{\underline{x}} = \lambda e^{\lambda t} \underline{c} = \lambda e^{\lambda t} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}$$

$$\text{so } \boxed{A\underline{c} = \lambda \underline{c}}$$

where $\underline{c}$ is unknown
and λ is unknown

The problem is then

$$[A - \lambda I] \underline{c} = \underline{0}$$

the eigenvalue (d) eigenvector ($\underline{c}$) problem
solved numerically

The solution of $\dot{\underline{x}} = A \underline{x}$ given A has unique eigenvalues

step 1 let $\underline{x} = E \underline{z}$ then $\dot{\underline{x}} = E \dot{\underline{z}}$

subs. into $\dot{\underline{x}} = A \underline{x}$ yields

$$E \dot{\underline{z}} = A E \underline{z}$$

step 2 Premult by E^{-1}

$$E^{-1} E \dot{\underline{z}} = E^{-1} A E \underline{z}$$

$$\dot{\underline{z}} = E^{-1} A E \underline{z}$$

step 3 Since $A = E \Lambda E^{-1}$

then

$$E^{-1} A E = \Lambda$$

proof: $E^{-1} A = E^{-1} (E \Lambda E^{-1})$
 $E^{-1} A E = E^{-1} (E \Lambda E^{-1}) E$
" $= \Lambda$

$$\dot{\underline{z}} = \Lambda \underline{z}$$

where $\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix}$

soln for $\underline{z}(t)$

$$\underline{z} = \begin{bmatrix} e^{\Lambda t} \end{bmatrix} \underline{z}(0)$$

where $e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots \\ & & & e^{\lambda_N t} \end{bmatrix}$

Why? one cannot analytically solve the characteristic equation for degree higher than 3

Look at a simple problem

$$\dot{\underline{x}} = \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}_A \underline{x}$$

$$[A - I\lambda] = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$[A - I\lambda] = \begin{bmatrix} 2-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}$$

$$\text{char. eqn: } \begin{vmatrix} 2-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 0(1) = 0$$

$$\lambda_1 = 2 \quad \left. \begin{array}{l} \lambda_1 = 2 \\ \lambda_2 = 1 \end{array} \right\} \text{ eigenvalues}$$

$$\lambda_2 = 1$$

Determine eigenvectors for each eigenvalue

$$\lambda_1 = 2 \quad \begin{bmatrix} 2-\lambda_1 & 1 \\ 0 & 1-\lambda_1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} c_2 = 0 \\ c_1 = \text{any value} \end{array}$$

$$\therefore \underline{c}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 1 \quad \begin{bmatrix} 2-\lambda_2 & 1 \\ 0 & 1-\lambda_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underline{c}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} c_1 + c_2 = 0 \\ \therefore c_2 = -c_1 \\ \text{however } c_1 = \text{any value} \end{array}$$

Consider

$$\frac{\bar{Y}}{\bar{U}} = \frac{20(s+2)}{s^2+4s+14}$$

Phasevariable representation

$$\frac{\bar{Y}}{\bar{U}} = \frac{\bar{X}}{\bar{U}} \frac{\bar{Y}}{\bar{X}}$$

if we define

$$\frac{\bar{Y}}{\bar{X}} = (s+2) \Rightarrow y = \dot{x} + 2x$$

$$\frac{\bar{X}}{\bar{U}} = \frac{20}{s^2+4s+14} \Rightarrow \ddot{x} + 4\dot{x} + 14x = u$$

define

$$x_1 \equiv x$$

$$x_2 \equiv \dot{x}$$

then $\boxed{\dot{x}_1 = x_2}$

and $\ddot{x} + 4\dot{x} + 14x = u \Rightarrow \boxed{\dot{x}_2 + 4x_2 + 14x_1 = u}$

$y = \dot{x} + 2x \rightarrow \boxed{y = x_2 + 2x_1}$

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -14 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

Finally $\dot{\underline{x}} = A \underline{x}$

then soln

$$\underline{x} = \underline{\Phi} \underline{x}(0)$$

where $\underline{\Phi} = E e^{At} E^{-1}$ "State transition matrix"

example

given $\dot{\underline{x}} = A \underline{x}$

then eigenvectors and values of A are computed as

$$\lambda_1 = 2; \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = -1; \underline{e}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Determine $\underline{\Phi}$

$$\begin{aligned} E &= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = [\underline{e}_1 \quad \underline{e}_2] \\ E^{-1} &= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix} \\ e^{At} &= \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \end{aligned} \left\{ \begin{aligned} \underline{\Phi} &= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{-1} \\ \underline{\Phi} &= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix} \end{aligned} \right.$$

$$\underline{\Phi} = \begin{bmatrix} e^{2t} & \frac{e^{-t} - e^{2t}}{2} \\ 0 & e^{-t} \end{bmatrix}$$

Using maxima to compute the STM

E: matrix([1,1],[0,z]);

EINV: E⁻¹;

EL: matrix([%e^(z*t), 0], [0, %e^(-1*t)]);

STM: E. EL. EINV;

quit();