Generation of Random Variates

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This paper is an invited presentation for the Parametric Technology Corporation, makers of MATHCAD

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Probability and Random Processes: Generation of Random Variates

by Professor Venkatarama Krishnan

Many good textbooks exist on probability and random processes written at the undergraduate level to the research level. However, there is no one handy and ready book that explains most of the essential topics, such as random variables and most of their frequently used discrete and continuous probability distribution functions, and random processes and their associated properties. A particular feature is the presentation of commonly occurring Fourier transforms where both the time and frequency functions are drawn to scale. Several probability tables with accuracy up to nine decimal places are provided in the appendices for quick reference.

Over 400 figures accompany the more than 300 examples given to help readers visualize how to solve the problems. In many instances, worked examples are solved with more than one approach to illustrate how different probability methodologies can work for the same problem.

This book is of particular value to students and professionals in electrical, computer, and civil engineering, physics, communications, biostatisicians, and applied mathematics.

Mathcad software was exclusively used for (i) drawing complicated composite graphs to scale, (ii) manipulating algebraic equations to obtain solutions, and (iii) constructing probability tables to nine places of decimal accuracy. The ease of use of the Mathcad software enabled that all algebraic calculations were verifiable with it.

You can find more information about the book or purchase it from Wiley at **http://www.wiley.com/WileyCDA/WileyTitle/productCd-0471703540,descCd-description.html**

Here's an excerpt from the book:

1. Rayleigh Variates from Gaussian Variates by Inverse Transformation

Generation of Gaussian Variates by Box-Mueller transformation

$N := 10000$ # Data points

$i := 0..N$

Creating two independent U(0,1) RV u and v

$u_i := \mathrm{rnd}(1)$ $\quad v_i := \mathrm{rnd}(1)$

$\mathrm{mean}(u) = 0.504$ $\quad \mathrm{var}(u) = 0.085$ $\quad \mathrm{mean}(v) = 0.5$ $\quad \mathrm{var}(v) = 0.083$

Transforming u and v to Gaussian RV a and b of nominal 0 mean and nominal unit variance

$a_i := \sqrt{-2\cdot\ln(u_i)}\cdot\cos(2\cdot\pi\cdot v_i)$ $\quad b_i := \sqrt{-2\cdot\ln(u_i)}\cdot\sin(2\cdot\pi\cdot v_i)$

$\mu a := \mathrm{mean}(a)$ $\quad \sigma a := \sqrt{\mathrm{var}(a)}$ $\quad \mu b := \mathrm{mean}(b)$ $\quad \sigma b := \sqrt{\mathrm{var}(b)}$

$\mu a = -0.014$ $\quad \sigma a = 0.995$ $\quad \mu b = 0.005$ $\quad \sigma b = 1.002$

Transforming a and b to x and y with exact 0 mean and unit variance Gaussian RV

$$x_i := \frac{a_i - \mu a}{\sigma a} \qquad y_i := \frac{b_i - \mu b}{\sigma b}$$

$$\text{mean}(x) = 0 \qquad \text{var}(x) = 1 \qquad \text{mean}(y) = 0 \qquad \text{var}(y) = 1$$

Plotting histograms

$M := 100$ No of Bins

$j := 0 .. M \qquad k := 0 .. M - 1$

$\text{int}_j := \frac{j}{12.5} - 4$ Start plot at Ð4 $\Delta := \frac{1}{12.5}$ Bin Interval $C := N \cdot \Delta$ Normalizing constant

$\Delta = 0.08 \qquad C = 800$

Calling histograms

$fX1 := \text{hist}(\text{int}, x) \qquad fY1 := \text{hist}(\text{int}, y)$

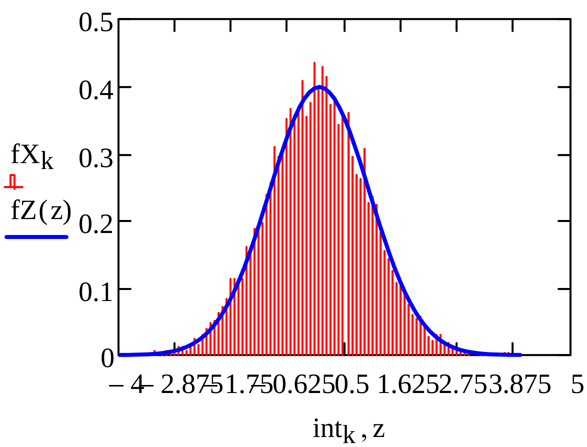
Standard Gaussian:

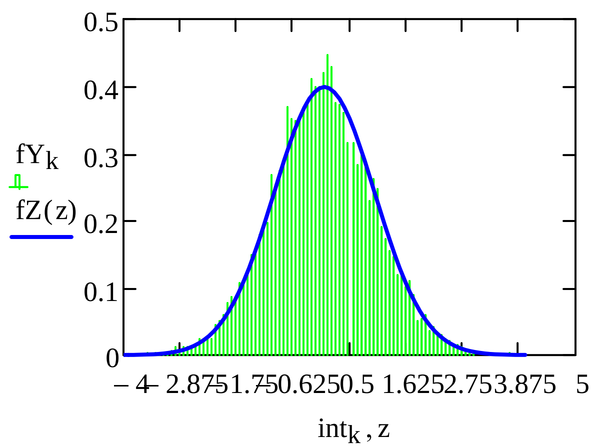
$z := -4, -3.9 .. 4$

Normaliizing

$$fX := \frac{fX1}{C} \qquad fY := \frac{fY1}{C} \qquad fZ(z) := \frac{1}{\sqrt{2 \cdot \pi}} \cdot \exp\left(\frac{-z^2}{2}\right)$$

Plotting Histograms and Standard Gaussian densities





Obtaining Rayleigh Variates from Gaussian

$$z_i := \sqrt{\left(x_i\right)^2 + \left(y_i\right)^2} \qquad w_i := \text{atan}\left(\frac{y_i}{x_i}\right)$$

Forming Rayleigh Variates

Plotting histograms

$M := 200$ No of Bins $\quad \Delta := \frac{1}{20}$ Bin Interval $\quad C := N \cdot \Delta$ Normalizing constant

$j := 0..M \qquad k := 0..M-1 \qquad \Delta = 0.05 \qquad C = 500$

$inte_j := \frac{j}{20}$ Start Rayleigh at 0 $\quad intr_j := \frac{j}{20} - 2$ Start Phase at Ð2

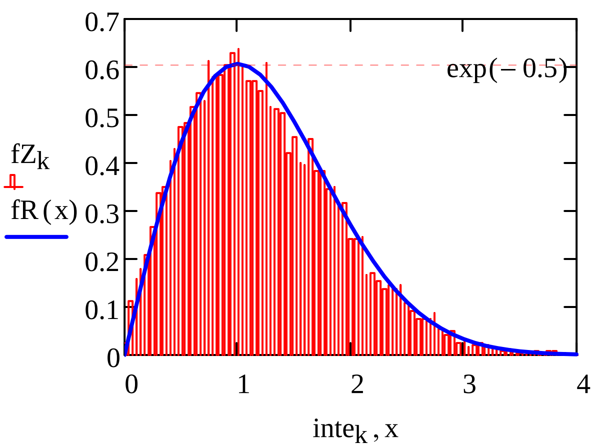
Calling histograms — True Rayleigh — True Phase

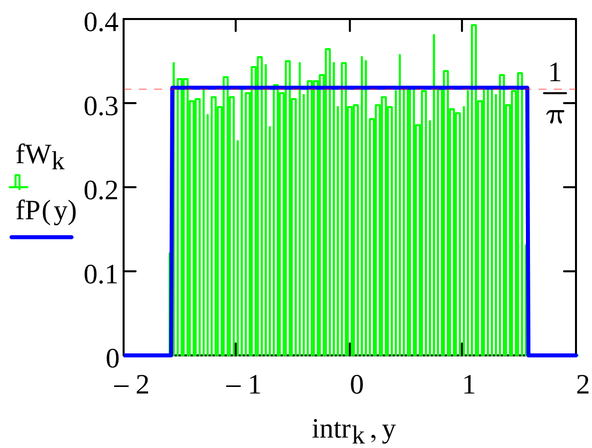
$a := hist(inte, z) \qquad b := hist(intr, w) \qquad x := 0, 0.1..4 \qquad y := -2, -1.99..2$

Normalizing

$$fZ := \frac{a}{C} \qquad fW := \frac{b}{C} \qquad fR(x) := x \cdot \exp\left(\frac{-x^2}{2}\right) \qquad fP(y) := \begin{cases} \frac{1}{\pi} & \text{if } \frac{-\pi}{2} < y \le \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

Plotting Histograms and True Rayleigh and Phase densities





2. Erlang Variates from Two Exponential Variates from Convolution

Generating Two Independent Exponential Variates from Inverse Transformation

$N := 10000$ # Data points

$i := 1..N$

$u_i := rnd(1) \qquad v_i := rnd(1)$ Generating 2 uniformly distributed random variates

$mean(u) = 0.499 \qquad mean(v) = 0.501 \qquad var(u) = 0.084 \qquad var(v) = 0.083$

$x_i := -3 \cdot \ln(u_i) \qquad y_i := -3 \cdot \ln(v_i)$ Transforming uniform RV to 2 exponential RV

$mean(x) = 2.987 \qquad var(x) = 8.63 \qquad mean(y) = 2.985 \qquad var(y) = 8.907$

Plotting Histograms

$M := 100$ No of Bins $\Delta := \frac{1}{10}$ Bin Interval $C := N \cdot \Delta$ Normalizing constant

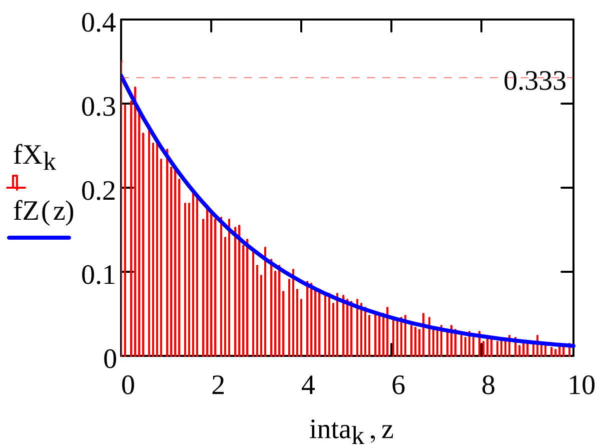
$j := 0..M$ $k := 0..M-1$ $\Delta = 0.1$ $C = 1000$

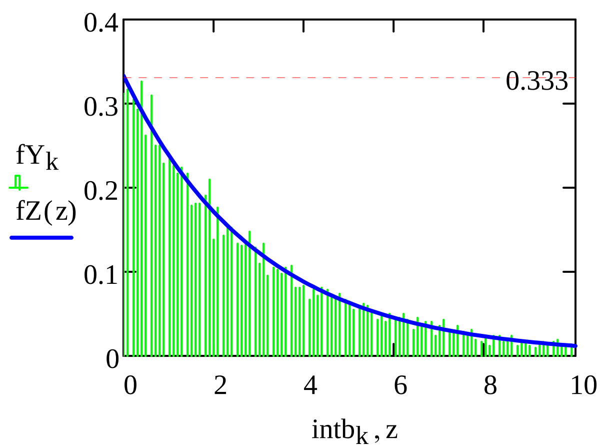
$inta_j := \frac{j}{10}$ Start at 0 $intb_j := \frac{j}{10}$ Star at 0 True Exponential Density $z := 0, 0.1..10$

$a := hist(inta, x)$ $b := hist(intb, y)$ Calling histograms $fZ(z) := \frac{1}{3} \cdot \exp\left(\frac{-1}{3} \cdot z\right)$

$fX := \frac{a}{C}$ $fY := \frac{b}{C}$ Normalizing

Plotting Histograms and True Exponential Densities





Generating Erlang Variates with Two degrees of freedom

$z_i := x_i + y_i$

$mean(z) = 5.972$ $var(z) = 17.523$

$M := 200$ No of bins $\Delta := \frac{1}{10}$ Bin Interval $C := N \cdot \Delta$ Normalizing constant

$j := 0..M$ $k := 0..M-1$ $\Delta = 0.1$ $C = 1000$

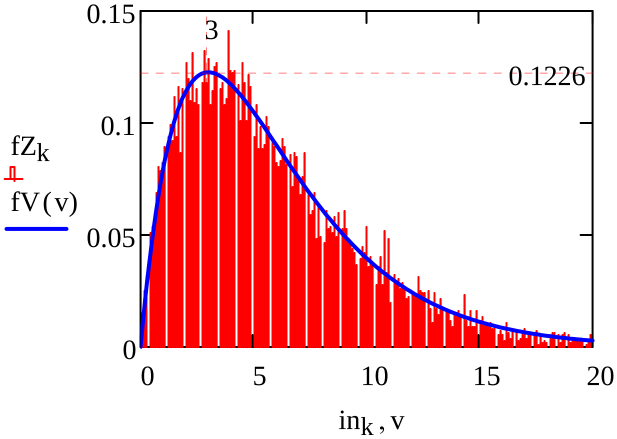
$in_j := \frac{j}{10}$ Making bin intervals = 0.1

Calling Histograms

True Erlang with 2 degrees of freedom $v := 0, 0.1..20$

$g := hist(in, z)$ $fZ := \frac{g}{C}$ Normalizing g $fV(v) := \left(\frac{1}{3}\right)^2 \cdot v \cdot \exp\left(\frac{-v}{3}\right)$

Plotting Histograms and True Erlang density



3. Beta Distribution Variates from Acceptance Rejection

Plotting Beta density

For Integer values of α and β, with $\alpha := 3$ and $\beta := 6$,fB(x) is given by,

General Beta density

$$fB(x) := \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)}\cdot x^{\alpha-1}\cdot(1-x)^{\beta-1}$$

$$fB(x) := 168\cdot x^2\cdot(1-x)^5$$

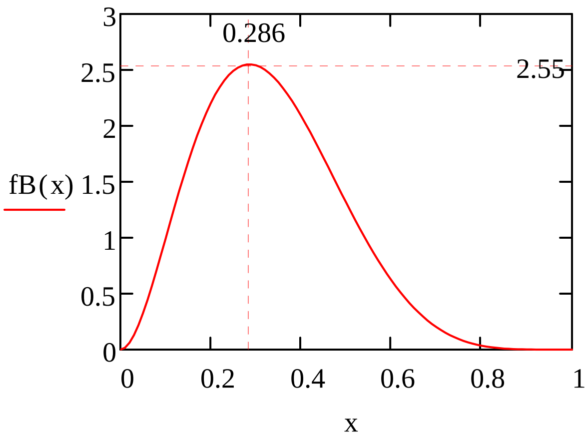
Maximum Value of Beta density occurs at $x0 := \frac{336}{1176} = 0.2857143$ and

the maximum value $fB(x0) = 2.549958 = \frac{300000}{117649}$

Plot of Beta Density $\quad x := 0, 0.01 .. 1$

$x0 = 0.28571 \qquad fB(x0) = 2.54996$

Plot of Beta Density $\beta(x, 3, 6)$



Acceptance Rejection Method of generating Beta Variates

$N := 10000$ Number of Data Points

$i := 0 .. N$

$y_i := \text{rnd}(1)$ $u_i := \text{rnd}(1)$ Generating 2 Uniform Random Variates

$\text{mean}(y) = 0.504$ $\text{var}(y) = 0.084$ $\text{mean}(u) = 0.497$ $\text{var}(u) = 0.083$

$g(\alpha) := 2.55$ Maximum value of fB(x)= $fB(x0) = 2.55$.
Set Bounding Function $g(\alpha) = 2.55 > fB(x)$.

$$x_i := \begin{vmatrix} y_i \ \text{ if } \ u_i \le \dfrac{168 \cdot (y_i)^2 \cdot (1 - y_i)^5}{2.55} \\ 0 \ \text{ otherwise} \end{vmatrix}$$ Setting the logic for accepting points

$\text{mean}(x) = 0.13$ $\text{var}(x) = 0.036$

$w := \text{sort}(x)$ Sorting x for eliminating intermediate 0 values

$w_{6077} = 0$ 6093 points are not accepted since they are 0

$w_{6078} = 0$ Finding number of accepted points $10000 - 6077 = 3923$

$w_{10000} = 0.833$ 3913 points go to make up x_i

.

$M := 100$ No of bins

$j := 0..M$

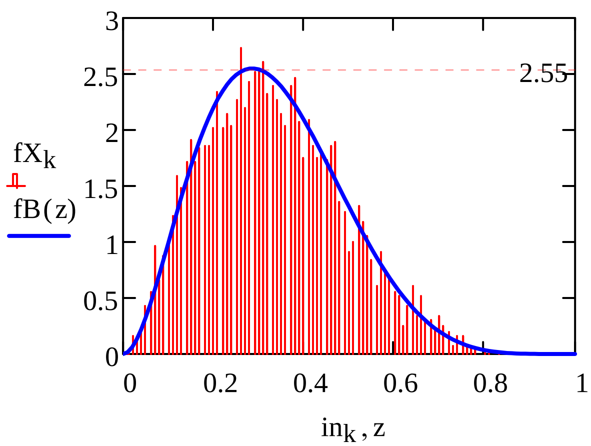
$k := 1..M-1$

$in_j := \frac{j}{M}$ Starting at 0

$b := \text{hist}(in, w)$ Calling Histogram Routine

$fX := \frac{b}{C}$ Normalizing g

Plotting Histogram and True Beta Density



$\Delta x := \frac{1}{M}$ $\Delta x = 0.01$ Bin Interval

$C := 3923 \cdot \Delta x$ $C = 39.23$ Set Normalization Constant

True Beta Density $z := 0, 0.01..1$

$f(x) := 168 \cdot z^2 \cdot (1-z)^5$

$\frac{N}{g(\alpha)} = 3921.569$ Accepted points ~ 3923

$N - \frac{N}{g(\alpha)} = 6078.431$ Rejected Points ~ 6077