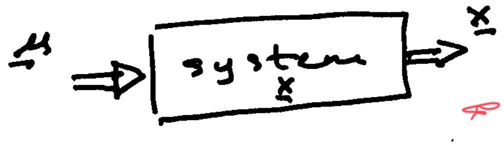
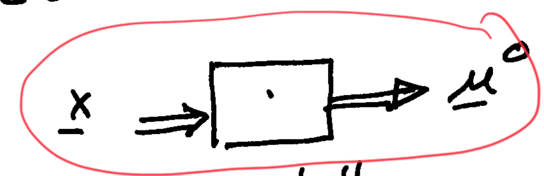
Optimal closed-loop control



① we have knowledge of $\underline{x}$ "obs"

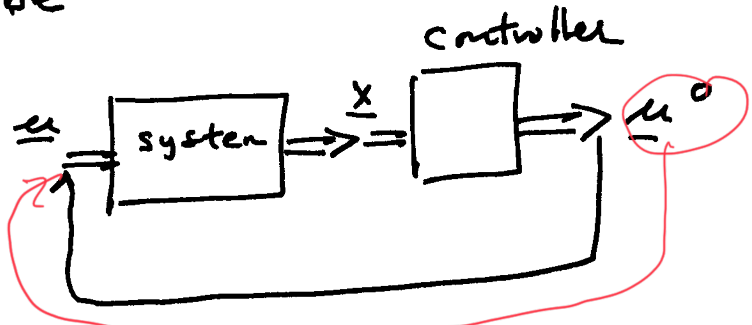
② any $\underline{x}$ is possible given the right $\underline{u}$ "cnt"

Objective is to find the controller such that



controller

The complete controlled system will be



~~I~~ In addition we will assume that $\underline{u}^o$ is linear function of $\underline{x}$

Problem statement

$$\min \quad PI = \int_{t}^{t_f} \left[\underline{x}^T Q \underline{x} + \underline{u}^T P \underline{u} \right] dt$$

subject to the constraint

$$\underline{\dot{x}} = A\underline{x} + B\underline{u}$$

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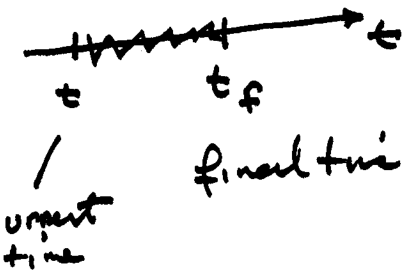
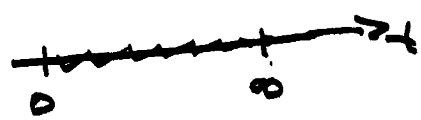
note new features

① introduction of the control time horizon

$$PI = \int_{t}^{t_f} [\;\;] dt$$



ie inf horizons



$$PI = \int_{0}^{\infty} [\;\;\;] dt$$

(2) Introduction quadratic form

$$\underline{x}^T Q \underline{x}$$

ie

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\boxed{\underline{x}^T Q \underline{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

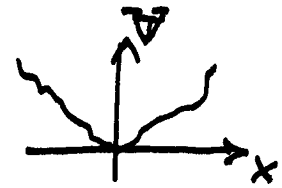
$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$$

$$\underline{x}^T Q \underline{x} = x_1(ax_1 + bx_2) + x_2(cx_1 + dx_2)$$

$$\boxed{\underline{x}^T Q \underline{x} = a x_1^2 + x_1 x_2 (b+c) + d \cdot x_2^2}$$

Prop of $\underline{x}^T Q \underline{x} = V$

① V is positive definite



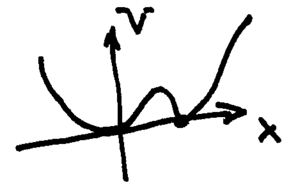
if $V > 0$ for all values of $\underline{x} \neq 0$

Therefore Q is symmetric and eigenvalue positive

① $Q_{ij} = Q_{ji}$

② $\lambda_i > 0$

② V is positive semidefinite



if $V \geq 0$ for all values of $\underline{x} \neq 0$

Therefore Q is symmetric and eigenvalue 0 or pos,

① $Q_{ij} = Q_{ji}$

② $\lambda_i \geq 0$

③ V negitive definite $Q_{ij} = Q_{ji}$; $\lambda_i < 0$

④ V negitive semidef $Q_{ij} = Q_{ji}$; $\lambda_i \leq 0$

⑤ If not ①-④ indefinite

Converting quadratic poly to symmetric matrix form

$$\cancel{\bar{V} = x_1^2 + a x_1 x_2 + 6 x_2^2}$$

$$\bar{V} = a x_1^2 + b x_1 x_2 + c x_2^2$$

$$\begin{array}{cc} & \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \end{array}$$

$$\bar{V} = a x_1^2 + 6 x_1 x_3 + d x_2^2 + e x_3^2$$

$$\begin{array}{cc} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} a & 0 & 6/2 \\ 0 & d & 0 \\ 6/2 & 0 & e \end{bmatrix} \end{array}$$

$$\underline{x}^T Q \underline{x}$$

$$\underline{x} = E \underline{z}$$

$$\underline{z}^T E^T Q E \underline{z}$$

$$A E = E \Lambda$$

$$A = E \Lambda E^{-1}$$

Hamiton - Jacobi Approach

$$PI = \int_{t}^{t_f} \underline{x}^T Q \underline{x} + \underline{u}^T P \underline{u} \, dt$$

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

assume Q is positive - semidef

P is positive def

$$\boxed{H = (\nabla V)^T [A\underline{x} + B\underline{u}] + \underline{x}^T Q \underline{x} + \underline{u}^T P \underline{u}}$$

Some useful identities

$$\hat{V} = \underline{a}^T B \underline{c}$$

$$\frac{\partial \hat{V}}{\partial \underline{a}} = (\cancel{B\underline{c})^T = \underline{c}^T} B^T \cdot B\underline{c}$$

$$\frac{\partial \hat{V}}{\partial \underline{c}} = (\underline{a}^T B)^T = B^T (\underline{a}^T)^T = B^T \underline{a}$$

Find the optimal input

$$\frac{\partial H}{\partial \underline{u}} = 0 = \left[(\nabla V)^T B \right]^T + P\underline{u} + (\underline{u}^T P)^T$$

$$0 = B^T(\nabla V) + P\underline{u} + P^T\underline{u}$$

note P is sym. $P = P^T$

$$0 = B^T(\nabla V) + 2P\underline{u}$$

solve for $\underline{u}$

$$\boxed{\underline{u}^o = -\frac{1}{2} P^{-1} \left[B^T \nabla V \right]}$$

Find H^o

$$H^o = H\Big|_{\underline{u} = \underline{u}^o}$$

$$H^o = (\nabla V)^T A \underline{x} + (\nabla V)^T B \left[P^{-1} B^T (\nabla V) \right] \frac{1}{2}(-1)$$

$$+ \underline{x}^T Q \underline{x}$$

$$+ \frac{1}{4} \left[P^{-1} B^T (\nabla V) \right]^T P \left[P^{-1} B^T (\nabla V) \right]$$

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side calc.

$$\frac{1}{4} \left[P^{-1} B^T (\nabla V) \right]^T = \frac{1}{4} \left[(B^T \nabla V)^T (P^{-1})^T \right]$$

$$= \frac{1}{4} \left[(\nabla V)^T B (P^{-1})^T \right]$$

note $(P^{-1})^T = P^{-1}$

$$= \frac{1}{4} \left[(\nabla V)^T B P^{-1} \right]$$

$$H^o = (\nabla V)^T A \underline{x} - \frac{1}{2} (\nabla V)^T B P^{-1} B^T (\nabla V) + \underline{x}^T Q \underline{x}$$

$$+ \frac{1}{4} (\nabla V)^T B P^{-1} \underbrace{P \, P^{-1}}_{1} B^T (\nabla V)$$

$$\boxed{H^o = (\nabla V)^T A \underline{x} - \frac{1}{4} (\nabla V)^T B P^{-1} B^T (\nabla V) + \underline{x}^T Q \underline{x}}$$

given HJ - equation

① $\dfrac{\partial V}{\partial t} + H^{o} = 0$

② $V = \underline{x}^T R \, \underline{x}$ R is positive def $R = R^T$

then

$$\left.\frac{\partial V}{\partial t}\right|_{\text{holding } \underline{x} \text{ cons}} = \underline{x}^T \dot{R} \, \underline{x}$$

$$(\nabla V) = R \underline{x} + R^T \underline{x} = (R + R^T) \underline{x} = 2 R \underline{x}$$

$\dfrac{\partial V}{\partial t} + H^{o} = 0$

$$\underline{x}^T \dot{R} \underline{x} + 2 \underline{x}^T R A \underline{x} - \underline{x}^T R B P^{-1} B^T R \underline{x} + \underline{x}^T Q \underline{x} = 0$$

$$\underline{x}^T \left[\dot{R} + 2RA - RBP^{-1}B^T R + Q \right] \underline{x} = 0$$

$$\therefore \quad \boxed{\dot{R} + 2RA - RBP^{-1}B^T R + Q = 0}$$

since R is to remain sym for all time take the sym. part of

$(2RA)$

$$(2RA)_{sym} = (RA) + (RA)^T$$

$$(2RA)_{sym} = RA + A^T R$$

$$\boxed{\dot{R} + RA + A^T R - R B P^{-1} B^T R + Q = 0}$$

$$\boxed{\text{where } R\Big|_{t=t_f} = [0]}$$

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the optimal input

$$\underline{u}^o = -\frac{1}{2} P^{-1} B (\nabla V) \quad \Leftarrow \quad \nabla V = 2R\underline{x}$$

$$\boxed{\underline{u}^o_{q} = -\frac{1}{2}\left[P^{-1} B R\right] \underline{x} (2)}$$