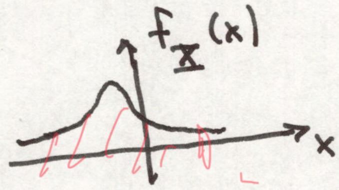
Some Probability

prob density function

$f_{\underline{X}}(x)$

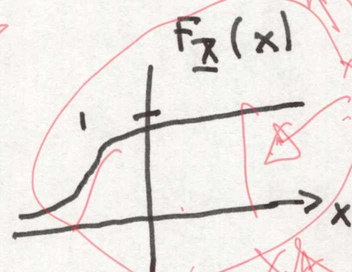


where $\displaystyle\int_{-\infty}^{\infty} f_{\underline{X}}(x)\,dx = 1$

$f_{\underline{X}}(x) \geq 0$

Cumulative Distribution

$x \to \infty$



$F_{\underline{X}}(x) = \text{Prob}(\underline{X} \leq x)$

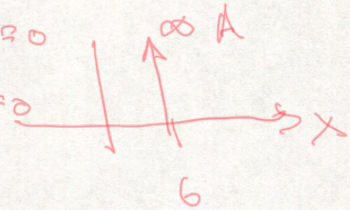
$F_{\underline{X}}(x) = \displaystyle\int_{-\infty}^{x} f_{\underline{X}}(q)\,dq$

$F_x(x) = \displaystyle\int_{-\infty}^{\infty} f_{\underline{X}}(x)\,dx$

$F_x(x) = \displaystyle\int_{-\infty}^{\infty} f_x(x)\,dx$

$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$

$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$



Consider

$$f_X(x) = \delta(x - 6)\,A$$

① if f_X is a pdf what is the value of A?

$A = ?$

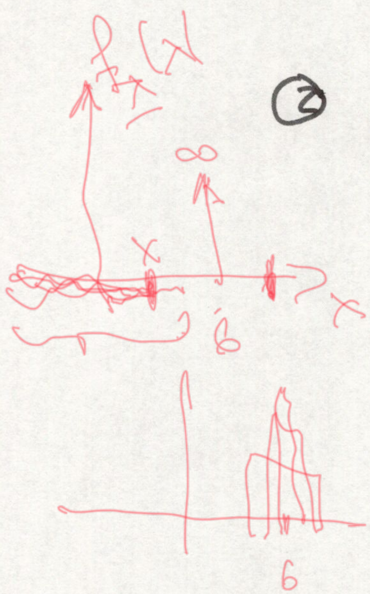
② what is $F_X(x)$

Soln

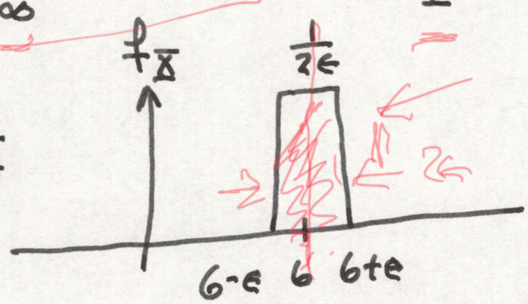
① $\int_{-\infty}^{\infty} f_X\,dx = 1$ ∴

$\int_{-\infty}^{\infty} A\,\delta(x-6)\,dx = A \implies \therefore A = 1$

② $F_X(x) = \int_{-\infty}^{x} \delta(q-6)\,dq = \begin{cases} 0 & x < 6 \\ \frac{1}{2} & x = 6 \quad ? \\ 1 & x > 6 \end{cases}$



~~why~~

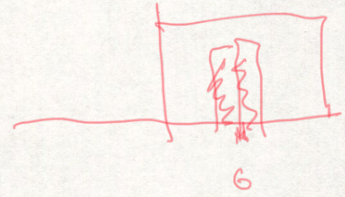


assuming $\delta(x-6)$ is sym about $x = 6$

Dirac delta

① $\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$ ✓

③ $\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$ ② $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$ ✓



Two variables

Joint density

$$f_{XY}(x, y)$$

where $\iint_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

Marginal density

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}\,dx$$

Product Rule

$$f_{XY} = \begin{cases} f_Y(y)\, f_X(x|y) \\ f_X(x)\, f_Y(y|x) \end{cases}$$

$$f_Y(y) f_X(x|y) = f_X(x) f_Y(y|x)$$

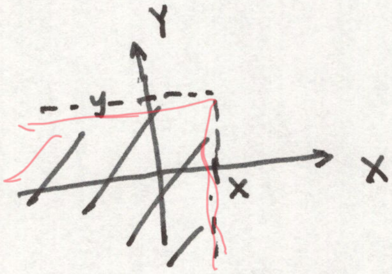
$$F_{XY} = \begin{cases} F_Y(y) \quad F_X(x, | Y \le y) \\ F_X(x) \quad \cancel{F_Y(X \le x | y)} \\ \quad F_Y(y | X < x) \end{cases}$$

$$f(x|y) = \frac{f(x)\, f_Y(y|x)}{f(y)}$$

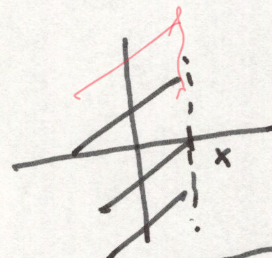
Consider the evaluation of

$$F_Y(y \mid X \le x) = \frac{F_{XY}(x,y)}{F_X(x)}$$

A. $$F_{XY} = \int_{-\infty}^{x}\left[\int_{-\infty}^{y} f_{XY}\,dy\right]dx$$



B. $$F_X(x) = \int_{-\infty}^{x}\left[\underbrace{\int_{-\infty}^{\infty} f_{XY}\,dy}_{f_X(x)}\right]dx$$



$$\boxed{F_Y(y \mid X \le x) = \frac{\int_{-\infty}^{x}\left[\int_{-\infty}^{y} f_{XY}\,dy\right]dx}{\int_{-\infty}^{x} f_X\,dx}}$$

$$g = \int_{a(x)}^{b(x)} f(q)\,dq$$

$$\frac{\partial g}{\partial x} = f(q)\Big|_{q=b(x)} - f(q)\Big|_{q=a(x)}$$

Question:

what is the density

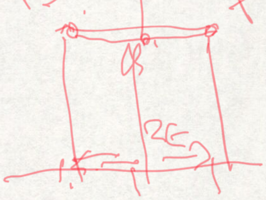
$$f_Y(y \mid X \le x)\ ?$$

Soln:

$$\frac{\partial}{\partial y} F_Y(y \mid X \le x) = \frac{\partial}{\partial y}\left[\frac{\int_{-\infty}^{x}\left[\int_{-\infty}^{y} f_{xy}\, dy\right] dx}{\int_{-\infty}^{x} f_x\, dx}\right]$$

$$\boxed{f_Y(y \mid X \le x) = \frac{\int_{-\infty}^{x} f_{XY}\, dx}{\int_{-\infty}^{x} f_X\, dx}}$$

$f(x-\epsilon)$ $f(x)$ $f(x+\epsilon)$



2ϵ

$x-\epsilon$ x $x+\epsilon$

$f(x)2\epsilon$

$f(x)$

$x-\epsilon$ x $x+\epsilon$

$$f_Y(y \mid x < X \le x+dx) = \frac{\int_{x}^{x+dx} f_{xy}\, dx}{\int_{x}^{x+dx} f_x\, dx}$$

$$\int_{a-\epsilon}^{a+\epsilon} f(x)\, dx \approx$$

$\epsilon \to 0$

$\cancel{\approx dx f(x)}$

Question

what is $f_Y(y \mid X = x)$

soln:

$$F_Y(y \mid x < X < x+dx) = \frac{\int_x^{x+dx} \left[\int_{-\infty}^{y} f_{XY}\, dy \right] dx}{\int_x^{x+dx} f_X\, dx}$$

$$\frac{\partial F_Y}{\partial y}(y \mid x < X < x+dx) = \frac{\int_x^{x+dx} f_{XY}\, dx}{\int_x^{x+dx} f_X\, dx}$$

using

$\cancel{\lim_{dx \to 0}^{x+dx} g(x)}$ $\lim_{dx \to 0} \int_x^{x+dx} g(x)\,dx \sim g(x)\,dx$

$$f_Y(y \mid x = X) = \lim_{dx \to 0} \frac{\int f_{XY}}{\int f_X} \sim \frac{f_{XY}\, \cancel{dx}}{f_X\, \cancel{dx}}$$

$$\boxed{f_Y(y \mid x = X) = \frac{f_{XY}}{f_X}}$$

Expectation

(a) given $x: f_{\underline{X}}(x)$; $x, y: f_{\underline{X}\underline{Y}}$

i) $E(g(x)) = \int_{-\infty}^{\infty} g(x) f_{\underline{X}}(x)\,dx$

ii) $E(g(x)|y) = \int_{-\infty}^{\infty} g(x) f_{\underline{X}\underline{Y}}(x|y)\,dx$

(b) Common usage

(i) $\boxed{\text{mean} = E(x) = \int_{-\infty}^{\infty} x f_{\underline{X}}\,dx = \bar{x}}$

(ii) variance $= E((x-\bar{x})^2) = \sigma^2$

$\sigma^2 = E(x^2 - 2\bar{x}x + \bar{x}^2)$

$\sigma^2 = E(x^2) - \underbrace{2\bar{x}E(x)}_{} + \bar{x}^2$

$E(x^2) \quad - 2\bar{x}^2 \quad + \bar{x}^2$

$$\boxed{\begin{array}{l}\sigma^2 = E(x^2) - \bar{x}^2 \\ \text{where } E(x^2) = \int_{-\infty}^{\infty} x^2 f_{\underline{X}}\,dx\end{array}}$$

xvu

$E(xa) = a\,E(x)$

$\sigma^2 = E((x-\bar{x})^2)$

$= E(x^2 - 2x\bar{x} - \bar{x}^2)$

$= E(x^2) - 2E(x\bar{x}) - E(\bar{x}^2)$

$= E(x^2) - 2\bar{x}\,\underset{\bar{x}}{E(x)} - \bar{x}^2$

$E(x^2) - x^2$

(c) covariance: $cov(x,y)$ given $f_{XY}(x,y)$

$$cov(x,y) = E\left((x-\bar{x})(y-\bar{y})\right)$$

$$cov(x,y) = E\left(xy - x\bar{y} - \bar{x}y + \bar{x}\bar{y}\right)$$

$$= E(xy) - \bar{x}\bar{y} - \bar{x}\bar{y} + \bar{x}\bar{y}$$

$$\boxed{\begin{aligned} &cov(x,y) = E(xy) - \bar{x}\bar{y} \\ &\text{where} \quad \bar{x} = \iint x f_{XY} \\ &\qquad\qquad \bar{y} = \iint y f_{XY} \\ &\qquad\quad E(xy) = \iint xy f_{XY} \end{aligned}}$$

(d) Multivariant

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \quad \underline{y} = \begin{bmatrix} y_1 \\ \\ y_N \end{bmatrix} \quad E(\underline{x}) = \overline{\underline{x}}$$

$$\text{cov}(\underline{x}, \underline{y}) = E\left[(\underline{x} - E(\underline{x})) (\underline{y} - E(\underline{y}))^T \right]$$

where

$$E(\underline{x}) = \overline{\underline{x}}$$

$$E(\underline{y}) = \overline{\underline{y}}$$

$$\text{cov}(\underline{x}, \underline{y}) = E\left[\underline{x}\,\underline{y}^T - E(\underline{x})\,\underline{y}^T - \underline{x}\,E(\underline{y})^T + E(\underline{x})E(\underline{y})^T \right]$$

$$= E(\underline{x}\,\underline{y}^T) - E(\underline{x})E^T(\underline{y}) - \cancel{E(\underline{x})E^T(\underline{y})} + \cancel{E(\underline{x})E^T(\underline{y})}$$

$$\text{cov}(\underline{x}, \underline{y}) = E(\underline{x}\,\underline{y}^T) - E(\underline{x})E^T(\underline{y})$$

$$\text{cov}(\underline{x}, \underline{y}) = E(x_i y_j) - (\overline{x}_i \overline{y}_j)$$

$$i = 1, N$$
$$j = 1, N$$

$$\text{cov}(\underline{x}, \underline{y}) = \begin{bmatrix} E(x_1 y_1) - \overline{x}_1 \overline{y}_1 & \cdots & E(x_1 y_N) - \overline{x}_1 \overline{y}_N \\ \vdots & & \\ & & E(x_N y_N) - \overline{x}_N \overline{y}_N \end{bmatrix}$$

Normal Distribution

(a) $x : N(\mu, \sigma^2)$ 1D

$$N(\mu, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{[x-\mu]^2}{2\sigma^2}}$$

pdf

μ mean

σ^2 variance

(b) $\underline{x} : N(\underline{\mu}, \Sigma)$

$$N(\underline{\mu}, \Sigma) = \frac{1}{(2\pi)^{D/2} |\Sigma|^{1/2}} e^{-[\underline{x}-\underline{\mu}]^T \Sigma^{-1} [\underline{x}-\underline{\mu}] \frac{1}{2}}$$

where $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_D \end{bmatrix}$ $\underline{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_D \end{bmatrix}$

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \cdots & \Sigma_{1D} \\ \vdots & & \vdots \\ \Sigma_{D1} & \cdots & \Sigma_{DD} \end{bmatrix}$$

$$|\Sigma| = \det[\Sigma]$$

Example; 2D-Gaussian

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

$$\operatorname{cov}(\underline{x}, \underline{x}) = \left[E(x_i x_j) - \mu_i \mu_j \right]$$

$\Sigma_{ij} = \Sigma_{ji} \quad i \neq j$

$$= \begin{bmatrix} E(x_1^2) - \mu_1^2 & E(x_1 x_2) - \mu_1 \mu_2 \\ E(x_1 x_2) - \mu_1 \mu_2 & E(x_2^2) - \mu_2^2 \end{bmatrix}$$

$$\left[\Sigma\right] = \operatorname{cov}(\underline{x}, \underline{x}) = \begin{bmatrix} \sigma_1 \sigma_1 & \sigma_1 \sigma_2 r \\ \sigma_1 \sigma_2 r & \sigma_2 \sigma_2 \end{bmatrix} = \Sigma$$

Σ_{11} Σ_{12} Σ_{21} Σ_{22}

where r is called the correlation

$$\left[\Sigma\right]^{-1} = \frac{1}{1 - r^2} \begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-r}{\sigma_1 \sigma_2} \\ \frac{-r}{\sigma_1 \sigma_2} & \frac{1}{\sigma_2^2} \end{bmatrix}$$

$$|\Sigma| = \sigma_1^2 \sigma_2^2 \left[1 - r^2\right]$$

$y : N(0, 1)$ $\qquad$ $x : N(\mu, \sigma^2)$

$$\boxed{a = E(x) = \mu}$$

$x = a + by$

$E(x) = \mu = E(a + by) = a + E(by)$

$= a + b\cancel{E(y)}$

pdf

$$f_{\underline{x}} = \frac{1}{(\sqrt{2\pi})^2 |\Sigma|^{1/2}} e^{-[x_1-\mu_1, x_2-\mu_2]\,\Sigma^{-1}\begin{bmatrix}x_1-\mu_1\\ x_2-\mu_2\end{bmatrix}\frac{1}{2}}$$

Marginals

$$f_{\underline{X}_1} = \int_{-\infty}^{\infty} f(\underline{x})\,dx_2 = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}}$$

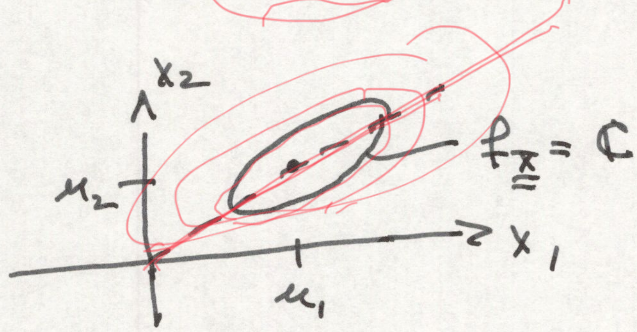
$$f_{\underline{X}_2} = \int_{-\infty}^{\infty} f(\underline{x})\,dx_1 = \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}}$$

Independence $r=0$ x_1 & x_2 are uncor.

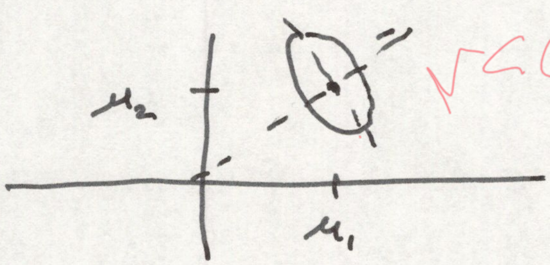
$$f_{\underline{X}} = f_{x_1} f_{x_2}$$

dependent

$r \neq 0$



$r>0$ pos corr.



$r<0$

$r<0$ neg corr

12

Generation of Gaussian rv

1-D

given $y: N_Y(0,1) \xrightarrow{\text{convert to}} x: N_X(\mu, \sigma^2)$

Linear transformation

$$x = a + by \quad \text{given } y: N(0,1)$$

(a) $E(x) = E(a+by) = a + bE(y) = \boxed{a = E(x)}$

(b) $E(x^2) = E((a+by)^2) = E(a^2 + 2aby + b^2y^2)$

$= a^2 + 2abE(y) + b^2E(y^2)$

note $E(y^2) - \cancel{E(y)^2}^{0} = 1$

$$\boxed{E(x^2) = a^2 + b^2}$$

$$\sigma^2 = E(x^2) - E(x)^2 = b^2$$

$$\mu = E(x) = a$$

$x =$ mean $+$ std y

$\boxed{x = \mu + \sqrt{\sigma^2}\, y}$ where $y: N(0,1)$

$x: N(\mu, \sigma^2)$

Multivar case

given $\underline{y} : N(\underline{0}, I)$

need to generate

$$\underline{x} : N(\underline{\mu}, \Sigma')$$

note $\Sigma_{ij} = \Sigma'_{ji}$

where

$$\underline{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_N \end{bmatrix} \quad \underline{x} = \begin{bmatrix} x_1 \\ \\ x_N \end{bmatrix}$$

$$\Sigma' = \begin{bmatrix} \Sigma'_{11} & \Sigma_{1N} \\ \Sigma_{N1} & \Sigma'_{NN} \end{bmatrix}$$

① Cholesky Factorization

$$\Sigma' = L L^T$$

Σ' $\quad \Sigma_{ij} = \Sigma'_{ji}$ if $i \neq j$

② Transform

$$\underline{x} = \underline{\mu} + L \underline{y}$$

where $\underline{x} : N(\underline{\mu}, \Sigma)$

2D-generation example

target $\underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ $\Sigma' = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2 r \\ \sigma_1\sigma_2 r & \sigma_2^2 \end{bmatrix}$

① Cholesky Factorization $\Sigma' = L L^T$

find L

$$\Sigma' = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} \\ 0 & l_{22} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11} l_{21} \\ l_{11} l_{21} & l_{22}^2 + l_{21}^2 \end{bmatrix}$$

match terms

$$l_{11}^2 = a \implies l_{11} = \sqrt{a}$$

$$l_{11} l_{21} = b \implies l_{21} = b/l_{11} = b/\sqrt{a}$$

$$l_{21}^2 + l_{22}^2 = c \implies l_{22} = \sqrt{c - l_{21}^2} = \sqrt{c - b^2/a}$$

$$l_{11} = \sigma_1^2$$

$$l_{21} = \frac{\sigma_1 \sigma_2 r}{\sigma_1} = \sigma_2 r$$

$$l_{22} = \sqrt{\sigma_2^2 - \frac{\sigma_1^2 \sigma_2^2 r^2}{\sigma_1^2}} = \sigma_2 \sqrt{1 - r^2}$$

$$L = \begin{bmatrix} \sigma_1 & 0 \\ \sigma_2 r & \sigma_2 \sqrt{1-r^2} \end{bmatrix}$$

The transformation

given $y : N(0,1)$

$$\boxed{\underline{x} = \underline{\mu} + L\, \underline{y}}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \cancel{\sigma_1} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \sigma_1 & 0 \\ \sigma_2 r & \sigma_2 \sqrt{1-r^2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$x_1 = \mu_1 + \sigma_1 y_1$$

$$x_2 = \mu_2 + \sigma_2 r y_1 + \sigma_2 \sqrt{1-r^2}\, y_2$$

where $y_1 : N(0,1)$

$y_2 : N(0,1)$