Chapter 7*

The Two-sided Z Transform

INTRODUCTION

Chapter 7 discusses the two-sided Z transform whose main application is to solve LTIC discrete systems with random or signal plus random inputs. This chapter is the transform analysis of the discrete material of Chapter 3.

The material is traversed by using what is now our standard treatment of any transform. The different stages are:

1. The transform is defined and a number of transforms are evaluated. We will utilize all our knowledge of one-sided transforms to help evaluate two-sided ones.
2. The properties and theorems are given and attention is focused on the transform of convolution and correlation summations.
3. The inverse transform is treated using previously mastered partial fraction techniques combined with table reference and also by using Laurent series plus residue theory from complex variables.
4. LTIC systems are solved with random or signal plus random inputs.

7-1 THE DEFINITION AND EVALUATION OF SOME TRANSFORMS

The two-sided or bilateral Z transform of a real discrete function $f(n)$ is defined as:

$$F_B(z) \triangleq \sum_{n=-\infty}^{\infty} f(n) z^{-n} \tag{7-1}$$

z) exists it will do so for all complex z in an annulus $\rho_1 < |z| < \rho_2$. Normally, script "B" is omitted and from the context it will be clear whether the one- -sided Z transform is being used. Alternate notations for $F(z)$ are)], and $\overline{f(n)}$. $f(n) \leftrightarrow F(z)$ is used to indicate the transform pair. number of transforms will now be evaluated and the relationship n $f(n)$ for positive and negative n to the annulus of convergence d.

LE 7-1

nd the two-sided Z transforms of the following functions and state the nulus of convergence:

) $f_1(n) = (-0.5)^n u(n)$
) $f_2(n) = (-0.5)^n u(-n)$
) $f_3(n) = 3(0.5)^n u(n) + 3^n u(-n)$
) $f_4(n) = 3(0.5)^n u(n) + 3^n u(n)$
) $f_5(n) = 3(0.5)^n u(-n) + 3^n u(-n)$
) $f_6(n) = A_1(\alpha)^n u(n) + A_2(\beta)^n u(-n)$ in general for all α and β.

ution

$$f_1(n) = (-0.5)^n u(n)$$

Therefore

$$F_1(z) = \frac{z}{z+0.5}, \qquad |z| > 0.5$$

$f_1(n)$ and $F_1(z)$ are shown in Figure 7-1(a), and since $f(n)$ is a causal function, the one- and two-sided transforms are identical.

$$f_2(n) = (-0.5)^n u(-n)$$

therefore

$$F_2(z) = 1 - 2z + 4z^2 - 8z^3 + \cdots$$

$$= \frac{1}{1+2z}, \qquad 2|z| < 1$$

$$= \frac{0.5}{z+0.5}, \qquad |z| < 0.5$$

$f_2(n)$ and $F_2(z)$ are plotted in Figure 7-1(b). It is easier to compare the transforms of $(-0.5)^n u(n)$ and $(-0.5)^n u(-n)$ when we write them as $1/(1 + 0.5z^{-1})$ and $1/[1 + (0.5z^{-1})^{-1}]$ and we may predict in general that:

$$a^n u(n) \leftrightarrow \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|$$

and

$$a^n u(-n) \leftrightarrow \frac{1}{1 - z/a} = \frac{-a}{z-a}, \qquad |z| < |a|$$

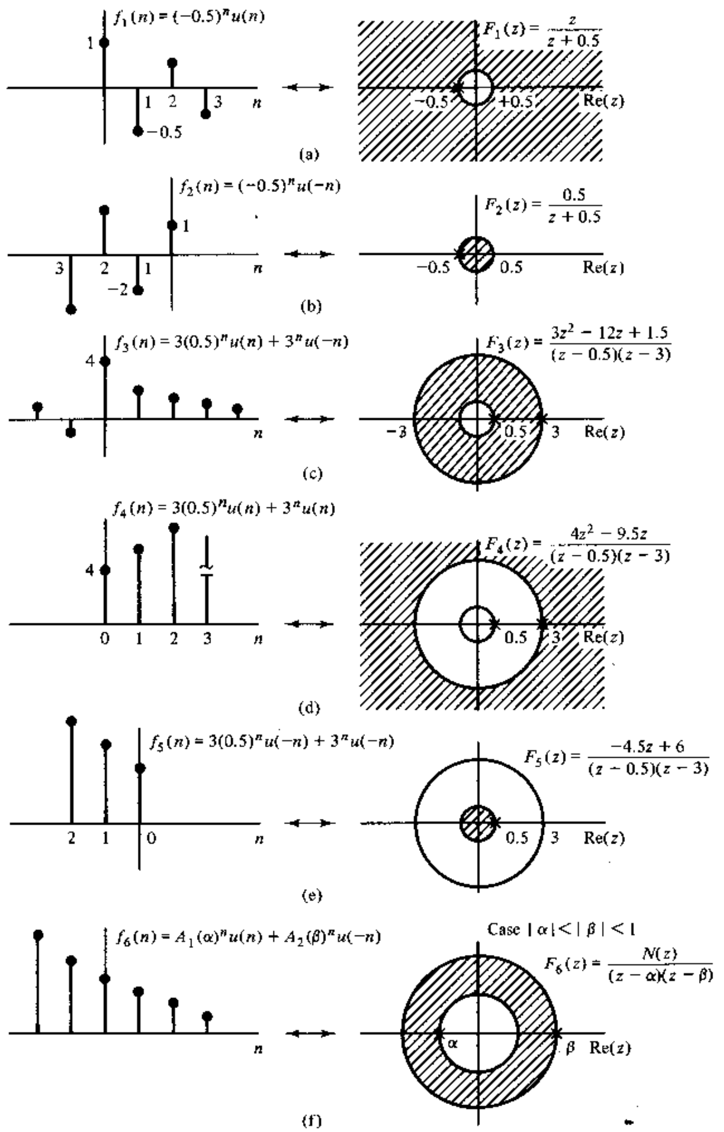


Figure 7-1 The discrete time functions of Example 7-1 and their two-sided transforms.

(c) $$f_3(n) = 3(0.5)^n u(n) + 3^n u(-n)$$

Therefore $$F_3(z) = \frac{3z}{z-0.5} + \frac{-3}{z-3}, \qquad |z| > 0.5 \cap |z| < 3$$
$$= \frac{3z^2 - 12z + 1.5}{(z-0.5)(z-3)}, \qquad 0.5 < |z| < 3$$

$f_3(n)$ and $F_3(z)$ are shown in Figure 7-1(c).

(d) $$f_4(n) = 3(0.5)^n u(n) + 3^n u(n)$$

therefore $$F_4(z) = \frac{3z}{z-0.5} + \frac{z}{z-3}, \qquad |z| > 0.5 \cap |z| > 3$$
$$= \frac{4z^2 - 9.5z}{(z-0.5)(z-3)}, \qquad |z| > 3$$

$f_4(n)$ and $F_4(z)$ are plotted in Figure 7-1(d), and since $f(n)$ is zero for $n < 0$, the annulus of convergence is outside all the poles.

(e) $$f_5(n) = 3(0.5)^n u(-n) + 3^n u(-n)$$
$$F_5(z) = \frac{-1.5}{z-0.5} + \frac{-3}{z-3}, \qquad |z| < 0.5 \cap |z| < 3$$
$$= \frac{-4.5z + 6}{(z-0.5)(z-3)}, \qquad |z| < 0.5$$

$f_5(n)$ and $F_5(z)$ are shown in Figure 5-1(e), and since the function is zero for $n > 0$ the annulus of convergence is inside all the poles.

(f) $$f_6(n) = A_1(\alpha)^n u(n) + A_2(\beta)^n u(-n)$$

therefore $$F_6(z) = \frac{A_1 z}{z-\alpha} - \frac{A_2\beta}{z-\beta}, \qquad |z| > |\alpha| \cap |z| < |\beta|$$

The Z transform will exist for all α, β such that $|\alpha| < |\beta|$. This general situation is demonstrated in Figure 7-1(f).

Reflection on Example 7-1 indicates that the behavior of $f(n)$ for $n < 0$ places an upper bound on $|z|$ and that the behavior of $f(n)$ for $n \geq 0$ places a lower bound on $|z|$. If $f(n)$ for both positive and negative n consists of products of exponents and polynomials of n (e.g., $f(n) = [(2^n + 3n(0.5)^n)u(n) + (2n+3)3^n u(-n)]$, then if $F(z)$ exists it will be the ratio of two equal order polynomials of z (if $f(0) \neq 0$).

The evaluation of Z transforms for any function of the form:

$$f(n) = f_1(n)u(n) + f_2(n)u(-n) \tag{7-2}$$

is straightforward for functions for which the one-sided Z transforms of $f_1(n)u(n)$ and $f_2(-n)u(n)$ are known:

$$Z[f(n)] = \sum_0^{\infty} f_1(n)z^{-n} + \sum_{-\infty}^{0} f_2(n)z^{-n}$$

$$= (f_1(0) + f_1(1)z^{-1} + \cdots)$$
$$+ (f_2(0) + f_2(-1)z + f_2(-2)z^2 + \cdots)$$
$$= Z[f_1(n)u(n)] + Z[f_2(-n)u(n)]|_{z=z^{-1}} \quad (7$$

The clear insightful understanding of:

$$Z[f_2(n)u(-n)] = Z[f_2(-n)u(n)]|_{z=z^{-1}} \quad (7\text{-}4$$

for $|z^{-1}| > \rho$ or $|z| < \rho^{-1}$ is very important.

We now find some two-sided Z transforms using a table of one-si transforms and Equation 7-3.

EXAMPLE 7-2

Find the Z transform of the following functions using Equation 7-3.

(a) $f_1(n) = a^n u(-n)$
(b) $f_2(n) = (-0.5)^n u(n) + (3 + n)(-3)^n u(-n)$

Solution

(a)

$$Z[a^n u(-n)] = Z[a^{-n}u(n)]|_{z=z^{-1}}$$
$$= \frac{z}{z - a^{-1}}\bigg|_{z=z^{-1}}$$
$$= \frac{z^{-1}}{z^{-1} - a^{-1}}$$
$$= \frac{-a}{z - a}, \quad |z^{-1}| > a^{-1}$$
$$= \frac{-a}{z - a}, \quad |z| < a$$

This agrees with our result from Example 7-1(b) when $a = -0.5$.

(b) Using Equation 7-3, we obtain:

$$Z[(-0.5)^n u(n) + (3 + n)(-3)^n u(-n)] = \frac{z}{z + 0.5} + Z[(3 - n)(-3)^{-n}u(n)]|_{z=z^{-1}}$$
$$= \frac{z}{z + 0.5} + Z\left[3\left(-\frac{1}{3}\right)^n u(n) - \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)^{n-1} n\, u(n)\right]\bigg|_{z=z^{-1}}$$
$$= \frac{z}{z + 0.5} + \frac{3z^{-1}}{z^{-1} + \frac{1}{3}} + \frac{1}{3}\frac{z^{-1}}{(z^{-1} + \frac{1}{3})^2}$$

$$= \frac{z}{z + 0.5} + \frac{9}{z + 3} + \frac{3z}{(z + 3)^2},$$
$$|z| > 0.5 \cap |z^{-1}| > \frac{1}{3}$$
$$= \frac{z^3 + 40.5z^2 + 24z}{(z + 0.5)(z + 3)^2}, \; 0.5 < |z| < 3$$

Finally, Table 7-1 gives a short list of two-sided Z transforms.

7-2 IMPORTANT THEOREMS OF BILATERAL Z TRANSFORMS

Table 7-2 lists some important theorems for two-sided Z transforms. Since the main application of two-sided Z transforms is to solve LTIC discrete systems with random or signal plus random inputs the convolution and correlation theorems are of the utmost importance and will now be proved and demonstrated.

EXAMPLE 7-3

Prove the convolution theorem and comment on the region of convergence.

TABLE 7-1 A TABLE OF TWO-SIDED Z TRANSFORMS

$f(n)$	$F(z) = \sum_{-\infty}^{\infty} f(n)z^{-n}$	Region of convergence
$a^n u(n)$	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$na^{n-1}u(n)$	$\frac{z}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
$n(n-1)a^{n-2}u(n)$	$\frac{2z}{(z-a)^3}$	$\lvert z\rvert > \lvert a\rvert$
$a^n u(-n)$	$\frac{-a}{z-a}$	$\lvert z\rvert < \lvert a\rvert$
$na^{n+1}u(-n)$	$\frac{-za^2}{(z-a)^2}$	$\lvert z\rvert < \lvert a\rvert$
$n(n+1)a^{n+2}u(-n)$	$\frac{-2z^2a^3}{(z-a)^3}$	$\lvert z\rvert < \lvert a\rvert$
$f_1(n)u(n) + f_2(n)u(-n)$	$Z[f_1(n)u(n)] + Z[f_2(-n)u(n)]\rvert_{z=z^{-1}}$	$\rho_1 < \lvert z\rvert < \rho_2$
$f_2(n)u(-n)$	$Z[f_2(-n)u(n)]\rvert_{z=z^{-1}}$	$\lvert z^{-1}\rvert > \rho_2' \cup \lvert z\rvert < \rho_2 = \frac{1}{\rho_2'}$

TABLE 7-2 TWO-SIDED Z TRANSFORM THEOREMS

Theorem	Time function	Two-sided Z transform	Region of convergence
	$f(n)$ $g(n)$	$F(z)$ $G(z)$	$\rho_{11} < \lvert z\rvert < \rho_{12}$ $\rho_{21} < \lvert z\rvert < \rho_{22}$
Linearity	$af(n) + bg(n)$	$aF(z) + bG(z)$	$\max(\rho_{11}, \rho_{21}) < \lvert z\rvert < \min(\rho_{12},$
Shifting	$f(n-k)$	$z^{-k}F(z)$	?
Convolution	$f(n)*g(n)$ $= \Sigma_{-\infty}^{\infty} f(p)g(n-p)$	$F(z)G(z)$	$\max(\rho_{11}, \rho_{21}) < \lvert z\rvert < \min(\rho_{12},$
Correlation	$f(n) \oplus g(n)$ $= \Sigma_{-\infty}^{\infty} f(p)g(p+n)$	$G(z)F(z^{-1})$	$\max(\rho_{21}, \rho_{12}^{-1}) < \lvert z\rvert < \min(\rho_{22},$
	$g(n) \oplus f(n)$ $f(n) \oplus f(n)$	$F(z)G(z^{-1})$ $F(z)F(z^{-1})$	? $\max(\rho_{11}, \rho_{12}^{-1}) < \lvert z\rvert < \min(\rho_{12},$

Note: The reader should fill in the two blanks marked by the question mark.

Solution. The convolution theorem states that if:

$$f(n) \leftrightarrow F(z), \qquad \rho_{f_1} < |z| < \rho_{f_2}$$

and

$$g(n) \leftrightarrow G(z), \qquad \rho_{g_1} < |z| < \rho_{g_2}$$

then

$$f(n)*g(n) \leftrightarrow F(z)G(z), \qquad \rho_1 < |z| < \rho_2$$

where ρ_1 and ρ_2 will be found. To prove this, we have:

$$f(n)*g(n) = \sum_{-\infty}^{\infty} f(p)g(n-p)$$

therefore

$$Z[f(n)*g(n)] = \sum_{n=-\infty}^{\infty}\left[\sum_{p=-\infty}^{\infty} f(p)g(n-p)\right]z^{-n}$$

Interchanging the order of summation, we obtain

$$Z[f(n)*g(n)] = \sum_{p=-\infty}^{\infty} f(p)\left(\sum_{n=-\infty}^{\infty} g(n-p)z^{-n}\right)$$

and letting $n - p = l$, we get:

$$Z[f(n)*g(n)] = \sum_{p=-\infty}^{\infty} f(p)\sum_{l=-\infty}^{\infty} g(l)z^{-p-l}$$

$$= \sum_{p} f(p)z^{-p}\sum_{l} g(l)z^{-l}$$

Therefore $Z[f(n)*g(n)] = F(z)G(z),$ for

$$(\rho_{f_1} < |z| < \rho_{f_2}) \cap (\rho_{g_1} < |z| < \rho_{g_2})$$

or

$$\max(\rho_{f_1}, \rho_{g_1}) < |z| < \min(\rho_{f_2}, \rho_{g_2}) \qquad (7\text{-}5$$

EXAMPLE 7-4

Find the Z transforms of the following and denote the region of convergence if the transform exists:

(a) $f_1(n) = (0.5)^n u(n)*(-0.6)^n u(-n)$
(b) $f_2(n) = u(n)*(0.5)^n u(-n)$
(c) $f_3(n) = (0.5)^{|n|}*u(n)$

Solution

(a)

$$F_1(z) = \frac{z}{z-0.5}\,\frac{0.6}{z+0.6}, \qquad |z| > 0.5 \cap |z| < 0.6$$

$$= \frac{0.6z}{(z-0.5)(z+0.6)}, \qquad 0.5 < |z| < 0.6$$

(b)

$$F_2(z) = \frac{z}{z-1}\,\frac{-0.5}{z-0.5}, \qquad |z| > 1 \cap |z| < 0.5 = \emptyset$$

therefore $F_2(z)$ does not exist.

(c) $(0.5)^{|n|} = 0.5^n u(n) + 0.5^{-n}u(-n) - \delta(n)$

We note the term $-\delta(n)$ is necessary since $(0.5)^n u(n)$ and $(0.5)^{-n}u(-n)$ each contribute a value of magnitude one at $n = 0$. So we must subtract $\delta(n)$.

$$Z[0.5^n u(n) + (0.5)^{-n}u(-n) - \delta(n)] = \frac{z}{z-0.5} + \frac{-2}{z-2} - 1, \qquad |z| > 0.5 \cap |z| < 2$$

$$= \frac{z^2 - 4z + 1}{(z-0.5)(z-2)} - \frac{z^2 - 2.5z + 1}{(z-0.5)(z-2)}$$

$$= \frac{-1.5z}{(z-0.5)(z-2)}, \qquad 0.5 < |z| < 2$$

Therefore

$$Z[(0.5)^{|n|}*u(n)] = \frac{-1.5z}{(z-0.5)(z-2)}\,\frac{z}{z-1}, \qquad 1 < |z| < 2$$

$$= \frac{-1.5z^2}{(z-0.5)(z-1)(z-2)}, \qquad 1 < |z| < 2$$

We now have two ways to evaluate discrete convolution summations:

1. directly from $\Sigma_p f(p)g(n-p)$ or
2. as the inverse transform of $F(z)G(z)$

EXAMPLE 7-5

Prove the correlation theorems:

(a) $x(n) \oplus y(n) \leftrightarrow Y(z)X(z^{-1})$
(b) $x(n) \oplus x(n) \leftrightarrow X(z)X(z^{-1})$

and discuss the regions of convergence.

Solution

(a) Now by definition:

$$Z\left[\sum_{k=-\infty}^{\infty} y(k)x(k-n)\right] = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} y(k)x(k-n)\right]z^{-n}$$

Interchanging the order of summation and using the substitution variable $k - n = p$, we obtain:

$$Z[x(n) \oplus y(n)] = \sum_{k=-\infty}^{\infty} y(k)\left[\sum_{n=-\infty}^{\infty} x(k-n)z^{-n}\right]$$

$$= \sum_{k=-\infty}^{\infty} y(k) \sum_{p=-\infty}^{\infty} x(p)z^{p-k}$$

$$= \sum_k y(k)z^{-k} \sum_p x(p)z^p$$

$$= Y(z)X(z^{-1}), \quad \text{for}$$

$$(\rho_{y_1} < |z| < \rho_{y_2}) \cap (\rho_{x_1} < |z^{-1}| < \rho_{x_2})$$

The annulus $(\rho_{x_1} < z^{-1} < \rho_{x_2})$ is equivalent to $\rho_{x_2}^{-1} < |z| < \rho_{x_1}^{-1}$.

Therefore $\quad Z[x(n) \oplus y(n)] = Y(z)X(z^{-1}), \quad$ for

$$(\rho_{y_1} < |z| < \rho_{y_2}) \cap (\rho_{x_2}^{-1} < |z| < \rho$$

or $\quad Z[x(n) \oplus y(n)] = Y(z)X(z^{-1}),$

$$\max(\rho_{y_1}, \rho_{x_2}^{-1}) < |z| < \min(\rho_{y_2}, \rho_{x_1}^{-1}) \quad (7\text{-}$$

(b) The Z transform of an autocorrelation function is a special case Equation 7-6:

$$Z[x(n) \oplus x(n)] = X(z)X(z^{-1}),$$

$$(\rho_{x_1} < |z| < \rho_{x_2}) \cap (\rho_{x_2}^{-1} < |z| < \rho_{x_1}^{-1}) \quad (7\text{-}$$

The region of convergence becomes, $\max(\rho_{x_1}, \rho_{x_2}^{-1}) < |z| < \min(\rho_{x_2}, \rho_{x_1}^{-1})$ and the transform exists if this annulus exists.

EXAMPLE 7-6

Find the Z transforms of the following correlation summations:

(a) $(-0.5)^n u(n) \oplus (-0.5)^n u(n)$
(b) $(0.5)^n u(n) \oplus 3^n u(-n)$
(c) $(0.5)^{|n|} \oplus u(n)$

Solution

(a) $(-0.5)^n u(n) \leftrightarrow \dfrac{z}{z+0.5}, \quad |z| > 0.5$

therefore

$$Z[(-0.5)^n u(n) \oplus (-0.5)^n u(n)] = \frac{z}{z+0.5}\,\frac{z^{-1}}{z^{-1}+0.5}$$

$$= \frac{z}{z+0.5}\,\frac{2}{z+2}, \quad 0.5 < |z| < 2$$

$$= \frac{2z}{(z+0.5)(z+2)}, \quad 0.5 < |z| < 2$$

(b)

$$Z[(0.5)^n u(n) \oplus 3^n u(-n)] = \frac{-3}{z-3}\,\frac{z^{-1}}{z^{-1}-0.5}$$

$$= \frac{+6}{(z-3)(z-2)}, \quad \max(0, 0) < |z| < \min(3, 2)$$

$$= \frac{+6}{(z-3)(z-2)}, \quad 0 < |z| < 2$$

(c) $(0.5)^{|n|} = (0.5)^n u(n) + (0.5)^{-n} u(-n) - \delta(n)$

$$= (0.5)^n u(n) + 2^n u(-n) - \delta(n)$$

Therefore

$$Z[(0.5)^{|n|}] = \frac{z}{z-0.5} + \frac{-2}{z-2} - 1$$

$$= \frac{-1.5z}{(z-0.5)(z-2)}, \quad 0.5 < |z| < 2$$

as in Example 7-4.

Therefore

$$Z[(0.5)^{|n|} \oplus u(n)] = \frac{z}{z-1}\frac{-1.5z^{-1}}{(z^{-1}-0.5)(z^{-1}-2)}$$

$$= \frac{z}{z-1}\frac{-1.5z}{0.5(z-2)^2(z-0.5)}$$

$$= \frac{-1.5z^2}{(z-0.5)(z-1)(z-2)}, \quad |z| > 1 \cap [0.5 < |z| < 2]$$

$$= \frac{-1.5z^2}{(z-0.5)(z-1)(z-2)}, \quad 1 < |z| < 2$$

If we need to find $0.5^{|n|} \oplus u(n)$, we now have two approaches:

1. evaluate $\Sigma_k\, y(k)x(k-n)$ or
2. find $Z^{-1}\,[-1.5z^2/(z-0.5)(z-1)(z-2)]$, where Z^{-1} indicates inverse Z transform.

7-3 THE INVERSE TWO-SIDED Z TRANSFORM

In this section two techniques for finding inverse two-sided transforms wi discussed:

1. the use of partial fraction expansions plus table reference
2. the classical evaluation using the theory of Laurent series and res theory

7-3-1 Inverse Transforms Using Partial Fractions

Given:

$$F(z) = \frac{N(z)}{D(z)} = \frac{b_m z^m + b_{m-1}z^{m-1} + \cdots + b_0}{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0}, \qquad \rho_1 < |z| < \rho_2$$

where the order of $N(z)$ is at most the same as that of $D(z)$ (is this common fo transforms?) we can expand $F(z)$ or $z^{-1}F(z)$ into partial fractions and fro table of one-sided transforms plus the fact that $Z[f_2(n)u(-n)]$ $Z[f_2(-n)u(n)]\,|_{z=z^{-1}}$ call off $f(n)$. A number of inverse transforms will now evaluated.

EXAMPLE 7-7

Find the inverse Z transforms of the following functions using part fractions:

(a) $F_1(z) = \dfrac{z^3 + 2z^2 + 2z}{(z+1)^2(z-2)}, \quad 1 < |z| < 2$

(b) $F_2(z) = \dfrac{z^3 + 2z^2 + 2z}{(z+1)^2(z-2)}, \quad |z| > 2$

(c) $F_3(z) = \dfrac{z^3 + 2z^2 + 2z}{(z+1)^2(z-2)}, \quad |z| < 1$

Solution

(a) Since the order of the numerator and denominator are the same, we express $F(z)/z$ in partial fractions:

$$\frac{F(z)}{z} = \frac{z^2 + 2z + 2}{(z+1)^2(z-2)}, \qquad 1 < |z| < 2$$

$$= \frac{A_1}{z+1} + \frac{A_2}{(z+1)^2} + \frac{A_3}{z-2}$$

$$A_2 = \frac{1 - 2 + 2}{-3} = -0.33,$$

$$A_3 = \frac{4 + 4 + 2}{9} = 1.11$$

and

$$A_1 = \left[\frac{d}{dz}\,\frac{z^2 + 2z + 2}{z-2}\right]_{z=-1}$$

$$= \frac{-3(0) - 1(1)}{9} = -0.11$$

Therefore

$$F(z) = \frac{-0.11z}{z+1} + \frac{-0.33z}{(z+1)^2} + \frac{1.11z}{z-2}, \quad 1 < |z| < 2$$

From our experience we know that the pole at $z = -1$ contributes to $f(n)$ for $n > 0$ and the pole at $z = 2$ contributes to $f(n)$ for $n < 0$

Therefore

$$f_1(n) = -0.11(-1)^n u(n) - 0.33n(-1)^{n-1}u(n) - 1.11(2)^n u(-n-1)$$

The inverse of $1.11z/(z-2)$, $|z| < 2$ requires some thought.

$$Z^{-1}\left[\frac{1.11}{z-2}\right] = -1.11\left(\frac{1}{2}\right)(2)^n u(-n) = g(n)$$

Therefore the inverse of $Z[1.11z/(z-2)]$ is $g(n+1) = -1.11\,(\frac{1}{2})(2^{n+1}u(-n-1)) = -1.11(2)^n u(-n-1)$, as was written in the expression for $f_1(n)$.

(b) $F_2(z) = \dfrac{z^3 + 2z^2 + 2z}{(z+1)^2(z-2)}, \quad |z| > 2$

Since all the poles are inside $|z| = 2$, then $f(n)$ is zero for $n < 0$

and $f_2(n) = -0.11(-1)^n u(n) - 0.33n(-1)^{n-1}u(n) + 1.11(2)^n u$

(c) $F_3(z) = \dfrac{z^3 + 2z^2 + 2z}{(z+1)^2(z-2)}, \qquad |z| < 1$

Since all the poles are outside $|z| = 1$, then $f_3(n)$ is zero for $n > 0$.

$$F_3(z) = \frac{-0.11z}{z+1} + \frac{-0.33z}{(z+1)^2} + \frac{1.11z}{z-2}, \qquad |z| < 1$$

$$f_3(n) = 0.11(-1)^n u(-n-1) + \overline{\frac{-0.33z}{(z+1)^2}} - 1.11(2)^n u(-n-1)$$

We must now discuss the inverse of $-0.33/(z+1)^2$. In general:

$$a^n u(-n) \leftrightarrow \frac{-a}{z-a}$$

Therefore
$$na^{n-1}u(-n) \leftrightarrow \frac{(z-a)(-1) + a(-1)}{(z-a)^2}$$
$$= \frac{-z}{(z-a)^2}$$

Using this relation, we have:

$$\frac{-0.33z}{(z+1)^2} \leftrightarrow 0.33n(-1)^{n-1}u(-n-1)$$

and
$$f_3(n) = [0.11(-1)^n + 0.33n(-1)^{n-1} - 1.11(2)^n]u(-n$$

Figure 7-2 parts (a) to (c) show $F(z)$ and its corresponding discrete-t function for this problem.

From Example 7-6 it can be seen that all the work required to eval inverse two-sided Z transforms by partial fractions was already mastered for one-sided case. The poles inside $|z|$ where $\rho_1 < |z| < \rho_2$ determine $f(n)$ for $n \geq$ whereas the poles outside $|z|$ determine $f(n)$ for $n < 0$. If $\alpha_1 \leq \rho_1$, then a te $A_1 z/(z - \alpha_1)$ contributes $(\alpha_1)^n u(n)$, whereas $A_1/(z - \alpha_1)$ contribu $(\alpha_1)^{n-1}u(n-1)$. If $\alpha_1 \geq \rho_2$, then a term $A_2/(z - \alpha_2)$ contributes $A_2(\alpha_2)^n u(-$ whereas $-A_2 z/(z - \alpha_2)$ contributes $A_2(\alpha_2)^{n+1}u(-n-1)$.

7-3-2 Inverse Two-sided Z Transforms Using Residues

The Appendix on complex variables summarizes the theory of Laurent series. the function $F(z) = N(z)/D(z)$ is expanded in a Laurent series in the region ρ_1 $|z| < \rho_2$ which represents an annulus between two consecutive poles, then:

$$F(z) = \sum_{n=-\infty}^{\infty} A_n z^n, \qquad \rho_1 < |z| < \rho_2 \qquad (7\text{-}8$$

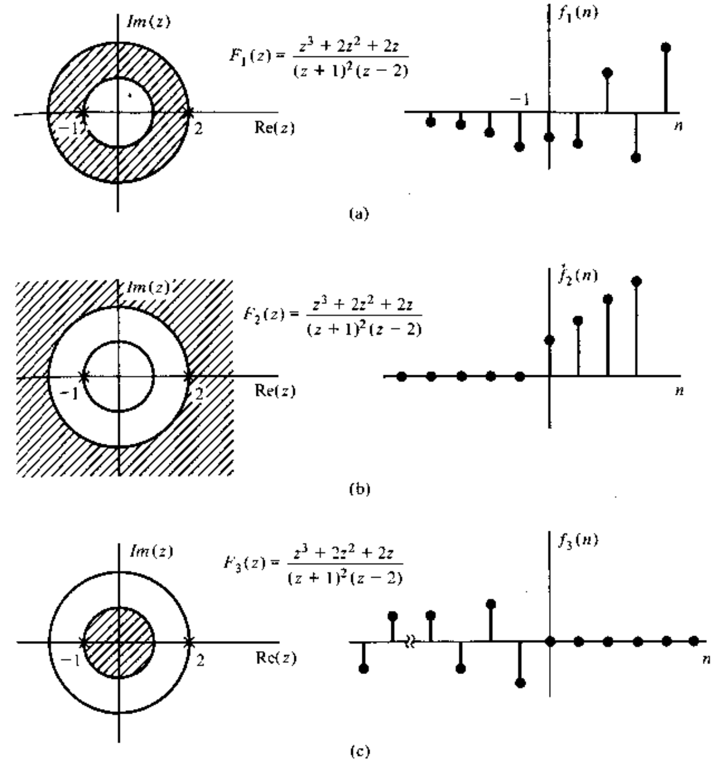


Figure 7-2 The Z transforms and their inverses for Example 7-7.

where the coefficients are given by:

$$A_n = \frac{1}{2\pi j} \oint_C \frac{F(z)}{z^{n+1}}\, dz \qquad (7\text{-}9)$$

where C is defined by $z(\theta) = \rho e^{j\theta}$, $0 < \theta \leq 2\pi$ with $\rho_1 < \rho < \rho_2$. Further, if the order of $N(z)$ is at most the order of $D(z)$, then the inside–outside theorem yields:

For $n > 0$

$$A_n = -\Sigma[\text{residues of the poles of } \frac{F(z)}{z^{n+1}} \text{ outside } C] \qquad (7\text{-}10)$$

For $n \leq 0$

$$A_n = \Sigma[\text{residues of the poles of } \frac{F(z)}{z^{n+1}} \text{ inside } C] \qquad (7\text{-}11)$$

As was seen in the Appendix the use of the inside–outside theorem allows us to avoid finding the residue of a higher-order pole at $z = 0$ for $n > 0$. We now must carefully adjust this theory of the Laurent series to evaluate inverse Z transforms. By definition:

$$Z[f(n)] = \cdots f(-n)z^n + \cdots + f(-1)z + f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots, \qquad \rho_1 < |z| < \rho_2$$

The Z transform is a Laurent series expansion where the Laurent coefficients are related to the discrete time values by:

$$f(-n) = A_n$$

or

$$f(n) = A_{-n}$$

Therefore, given:

$$Z[f(n)] = F(z), \qquad \rho_1 < |z| < \rho_2$$

we have:

$$f(n) = A_{-n} = \frac{1}{2\pi j}\oint \frac{F(z)}{z^{-n+1}}\,dz = \frac{1}{2\pi j}\oint z^{n-1}F(z)\,dz \qquad (7\text{-}$$

and from Equations 7-10 and 7-11 we obtain:

For $n \geq 0$

$$f(n) = \frac{1}{2\pi j}\oint_C z^{n-1}F(z)\,dz = \Sigma\,[\text{residues of the poles of } z^{n-1}F(z) \text{ inside } C] \qquad (7\text{-}$$

For $n < 0$

$$f(n) = \frac{1}{2\pi j}\oint_C z^{n-1}F(z)\,dz = -\Sigma\,[\text{residues of the poles of } z^{n-1}F(z) \text{ outside } C] \qquad (7\text{-}1$$

since $z^{n-1} = 1/z^{|n|+1}$ causes the order of the denominator to be more than o higher than the numerator and the inside–outside theorem may be used.

Summarizing, we conclude:

If

$$Z[f(n)] = F(z) = \frac{N(z)}{D(z)}, \qquad \rho_1 < |z| < \rho_2$$

and the order of $N(z)$ is at most equal to the order of $D(z)$, then the inverse transform:

$$f(n) = A_n = \frac{1}{2\pi j}\oint_C z^{n-1}F(z)\,dz$$

is:

for $n \geq 0$

$$f(n) = \Sigma\,[\text{residues of the poles of } z^{n-1}F(z) \text{ inside } C]$$

for $n < 0$

$$f(n) = -\Sigma\,[\text{residues of the poles of } z^{n-1}F(z) \text{ outside } C]$$

where C is defined by $z(\theta) = \rho e^{j\theta}$, $\rho_1 < \rho < \rho_2$.

We now find some inverse two-sided Z transforms using residue theory.

EXAMPLE 7-8

Find the inverse Z transforms of the following functions using residue theory:

(a) $F_1(z) = \dfrac{z^3 + 2z^2 + 2z}{(z+1)^2(z-2)}, \qquad 1 < |z| < 2$

(b) $F_2(z) = \dfrac{z^3 + 2z^2 + 2z}{(z+1)^2(z-2)}, \qquad |z| > 2$

(c) $F_3(z) = \dfrac{z^3 + 2z^2 + 2z}{(z+1)^2(z-2)}, \qquad |z| < 1$

Solution. We are now finding by residue theory the inverse transforms of the same functions whose inverses were found by partial fraction theory in Example 7-7.

(a) Figure 7-2(a) showed a pole zero diagram for $F_1(z)$.

Therefore

$$f_1(n) = \frac{1}{2\pi j}\oint_C z^{n-1}\frac{z^3 + 2z^2 + 2z}{(z+1)^2(z-2)}\,dz$$

For $n \geq 0$

$$f_1(n) = [\text{residue of the second-order pole at } z = -1]$$

$$= \frac{d}{dz}\left[\frac{z^n(z^2+2z+2)}{(z-2)}\right]_{z=-1}$$

$$= \left.\frac{[(z-2)[nz^{n-1}(z^2+2z+2) + z^n(2z+2)] - z^n(z^2+2z+2)(1)]}{(z-2)^2}\right|_{z=-1}$$

$$= \frac{1}{9}\{-3n(-1)^{n-1}(1) + (-1)^n(0)] - (-1)^n(1)\}$$

$$= -0.33n(-1)^{n-1} - 0.11(-1)^n$$

We note when $n = 0$ the pole at $z = 0$ has a zero residue:

For n < 0

$$f_1(n) = -\ [\text{residue of the pole at } z = 2]$$
$$= -\left.\frac{z^n(z^2 + 2z + 2)}{(z+1)^2}\right|_{z=2}$$
$$= -2^n\left(\frac{10}{9}\right)$$
$$= -1.11(2)^n$$

Summarizing the inverse transform yields:

$$f_1(n) = [-0.11(-1)^n - 0.33n(-1)^{n-1}]u(n) - 1.11(2)^n u(-n - 1)$$

This result agrees with part (a) of Example 7-7 and was shown in Fi 7-2(a).

(b) $$f_2(n) = \frac{1}{2\pi j}\oint_{|z|=\rho} z^n \frac{z^2 + 2z + 2}{(z+1)^2(z-2)}\,dz, \qquad \rho > 2$$

For n ≥ 0

$$f_2(n) = \Sigma\ [\text{residues of the poles at } z = -1 \text{ and } z = +2]$$
$$= [-0.11(-1)^n - 0.33n(-1)^{n-1} + 1.11(2)^n]$$

For n < 0

Since there are no poles of $F(z)z^{n-1}$ outside C, then:

$$f_2(n) = 0$$

Finally, the inverse transform is:

$$f_2(n) = [-0.11(-1)^n - 0.33n(-1)^{n-1} + 1.11(2)^n]u(n)$$

This agrees with Example 7-7(b), which is shown in Figure 7-2(b).

(c) $$f_3(n) = \frac{1}{2\pi j}\oint_{|z|=\rho} z^n \frac{z^2 + 2z + 2}{(z+1)^2(z-2)}\,dz, \qquad \rho < 1$$

For n ≥ 0

Since there are no poles inside C then $f_3(n) = 0$

For n < 0

$$f_3(n) = -\Sigma\ [\text{residues of the poles at } z = -1 \text{ and } z = 2]$$
$$= [0.11(-1)^n + 0.33n(-1)^{n-1} - 1.11(2)^n]$$

The inverse transform is:

$$f_3(n) = [0.11(-1)^n + 0.33n(-1)^{n-1} - 1.11(2)^n]u(-n - 1)$$

This agrees with Example 7-7(c) and is plotted in Figure 7-2(c).

7-3-3 Complex Convolution

In Chapter 5 we discussed complex convolution when finding the Laplace transform of the product of continuous functions. We now consider complex convolution for the product of discrete functions.

EXAMPLE 7-9

Prove:

$$K(z) = Z[f(n)g(n)] = \frac{1}{2\pi j}\oint_C F(p)G\left(\frac{z}{p}\right)p^{-1}\,dp$$
$$= F(z) * G(z)$$

given: $f(n) \leftrightarrow F(z), \quad \rho_{f1} < \rho < \rho_{f2}$

and $g(n) \leftrightarrow G(z), \quad \rho_{g1} < \rho < \rho_{g2}$

Pay particular attention to the restrictions on C and the region of convergence for $K(z)$.

Solution. Before starting our proof, we note that if $F(z)$ converges $\rho_{f1} < \rho < \rho_{f2}$ and $G(z)$ converges $\rho_{g1} < \rho < \rho_{g2}$, then $k(n) = f(n)g(n)$ must have a Z transform that converges $\rho_{f1}\rho_{g1} < \rho < \rho_{f2}\rho_{g2}$ [think carefully about this]. By definition:

$$K(z) = \sum_{n=-\infty}^{\infty} f(n)g(n)z^{-n}$$
$$= \sum_{-\infty}^{\infty} g(n)\left[\frac{1}{2\pi j}\oint_{C_1} F(p)p^{n-1}\,dp\right]z^{-n}$$

Now assuming it is permissible to interchange the order of summation and integration, we obtain:

$$K(z) = \frac{1}{2\pi j}\oint_{C_1} F(p)\left[\sum_{n=-\infty}^{\infty} g(n)\left(\frac{z}{p}\right)^{-n} p^{-1}\right]dp$$
$$= \frac{1}{2\pi j}\oint_C F(p)G\left(\frac{z}{p}\right)p^{-1}\,dp$$
$$= F(z) * G(z)$$

Now we must carefully discuss $C = \rho_k e^{j\phi}$. First, ρ_k must satisfy $\rho_{f1} < \rho_k < \rho_{f2}$. Also for any z such that $\rho_{f1}\rho_{g1} < |z| < \rho_{f2}\rho_{g2}$, we must have $\rho_{g1} < |z|$

$\div \rho_k < \rho_{g2}$. The solution of an actual problem will make us appreciate thes restrictions.

EXAMPLE 7-10

Consider finding $Z[f(n)g(n)]$ where $f(n) = 2^n u(-n) + u(n)$ and $g(n)$ $u(-n) + 0.5^n u(n)$ by complex convolution.

(a) Find the annulus of convergence for which $F(z)*G(z)$ exists.
(b) Sketch a pole zero diagram showing the poles of $F(p)G(z/p)p^{-1}$ a indicate where C is constrained in the p plane.
(c) Evaluate $K(z)$.

Solution

(a)

$$F(z) = Z[2^n u(-n) + u(n)]$$

$$= \frac{-2}{z-2} + \frac{z}{z-1}, \qquad 1 < |z| < 2$$

$$G(z) = Z[u(-n) + 0.5^n u(n)]$$

$$= \frac{-1}{z-1} + \frac{z}{z-0.5}, \qquad 0.5 < |z| < 1$$

$K(z) = F(z)*G(z)$ will exist for $0.5 < |z| < 2$ as is easily seen by find $f(n)g(n)$.

(b) With some work:

$$K(z) = \frac{1}{2\pi j} \oint_C \frac{p^2 - 4p + 2}{(p-2)(p-1)} \frac{p^2 - 4pz + 2z^2}{(p-z)(p-2z)p} dp$$

We note $K(z)$ has poles at $p = 0, 1, z, 2$, and $2z$, and we must satisfy t conditions for ρ_k in C defined by $\rho_k e^{j\phi}$, $0 < \phi \leq 2\pi$; first, $1 < \rho_k < 2$, a second, $|z| < \rho_k < 2|z|$ where $0.5 < |z| < 2$ from part (a). This requi the pole at $p = z$ is always inside the pole at $p = 2$ and the pole at $p =$ is always outside the pole at $p = 1$. Therefore the contour $\rho_K e^{j\phi}$ alwa has the poles at $p = 0$, $p = 1$, and $p = z$ inside it and the poles at $p =$ and $p = 2$ outside. The pole zero diagram is shown for different cases Figure 7-3. These moving poles at $p = z$ and $p = 2z$ in the p plane a tricky to visualize.

(c) The direct evaluation of:

$$K(z) = \frac{1}{2\pi j} \oint_C \frac{(p^2 - 4p + 2)(p^2 - 4pz + 2z^2)}{p(p-1)(p-z)(p-2)(p-2z)} dp$$

$$= \sum [\text{residues at } p = 0, 1, \text{and } z]$$

is very messy and so we will handle it in simple parts.

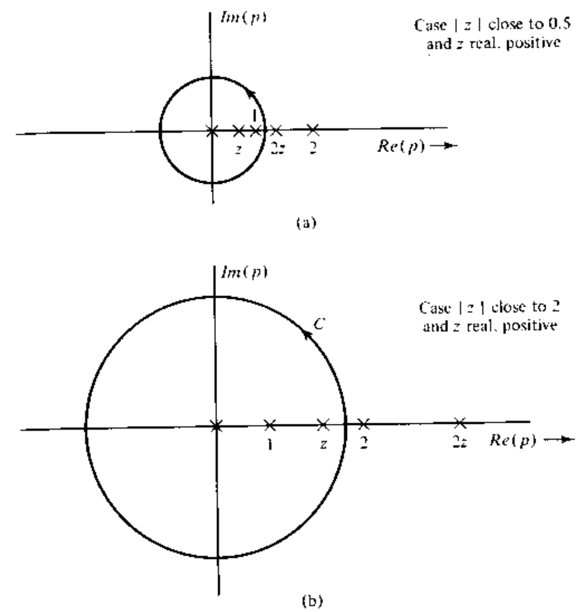


Figure 7-3 Pole zero diagrams for $F(p)G(z/p)p^{-1}$ for Example 7-10.

$$K(z) = \frac{1}{2\pi j} \oint_C \left(\frac{-2}{p-2} - \frac{p}{p-1}\right)\left(\frac{p}{p-z} - \frac{2z}{p-2z}\right)\frac{1}{p} dp$$

$$= \frac{1}{2\pi j} \oint_C \left[\frac{-2}{(p-2)(p-z)} + \frac{p}{(p-1)(p-z)} - \frac{2z}{(p-1)(p-2z)} + \frac{4z}{(p-2)(p-2z)p}\right] dp$$

$$= \frac{-2}{z-2} + 1 + \frac{z}{z-0.5} + 1$$

$$= \frac{3z^2 - 9z + 3}{(z-0.5)(z-2)}, \qquad 0.5 < |z| < 2$$

As a check we find $Z[f(n)g(n)]$ directly.

$$Z[2^n u(-n) + u(n)][u(-n) + 0.5^n u(n)]$$

$$= Z[2^n u(-n) + 2\delta(n) + 0.5^n u(n)]$$

therefore $$K(z) = \frac{-2}{z-2} + 2 + \frac{z}{z-0.5}, \qquad |z| < 2 \cap |z| > 0.5$$

This checks with our previous result. It is important when finding $f(n)g(n)$ to note that $u(n)u(-n) = \delta(n)$ and not 0.

7-4 LINEAR SYSTEMS WITH RANDOM AND SIGNAL PLUS NOISE INPUTS

In Chapter 3 we found that the output autocorrelation function and t cross-correlation of the input with the output when the input to a LTIC discre system is a noise waveform with autocorrelation function $R_{xx}(n)$ were:

$$R_{yy}(n) = C_{hh}(n)*R_{xx}(n) \quad (7\text{-}1$$

$$R_{xy}(n) = h(n)*R_{xx}(n) \quad (7\text{-}16$$

or

$$= R_{xx}(n) \oplus h(n) \quad (7\text{-}16$$

and

$$R_{yx}(n) = R_{xy}(-n) \quad (7\text{-}1$$

$$= h(n) \oplus R_{xx}(n) \quad (7\text{-}1$$

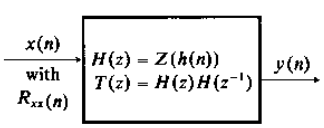
These results are shown schematically in Table 7-3(a).

Let us denote $Z[R_{xx}(n)]$ by $S_{xx}(z)$, $Z[R_{xy}(n)]$ by $S_{xy}(z)$, $Z[R_{yx}(n)]$ $S_{yx}(z)$, $Z[R_{yy}(n)]$ by $S_{yy}(z)$, and $Z[C_{hh}(n) = h(n) \oplus h(n)]$ by $T(z)$. We c $S_{xx}(z)$ the power spectral density of $x(n)$, $S_{xy}(z)$ the cross-spectral density $x(n)$ with $y(n)$, $S_{yx}(z)$ the cross-spectral density of $y(n)$ with $x(n)$, $S_{yy}(z)$ t power spectral density of $y(n)$, and $T(z)$ the power transfer function.

Using the convolution and correlation theorems, we find that Equati

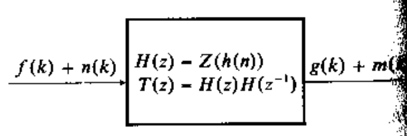
TABLE 7-3

(a)



Case $x(n)$ random

(b)



Case $f(k)$ deterministic, $n(k)$ zero-mean random and uncorrelated

Case $x(n)$ random	Case $f(k)$ deterministic, $n(k)$ zero-mean random and uncorrelated
Time-Domain Results from Chapter 3	**Time-Domain Results from Chapter 3**
$R_{yy}(n) = C_{hh}(n)*R_{xx}(n)$	$g(k) = f(k)*h(k)$
$R_{xy}(n) = h(n)*R_{xx}(n)$	$R_{mm}(k) = C_{hh}(k)*R_{nn}(k)$
$R_{yx}(n) = R_{xy}(-n)$	$R_{nm}(k) = h(k)*R_{nn}(k)$
	$R_{mn}(k) = R_{nm}(-k)$
Transform Results	**Transform Results**
$S_{yy}(z) = T(z)S_{xx}(z)$	$G(z) = F(z)H(z)$
$S_{xy}(z) = H(z)S_{xx}(z)$	$S_{mm}(z) = T(z)S_{nn}(z)$
$S_{yx}(z) = H(z^{-1})S_{xx}(z)$	$S_{nm}(z) = H(z)S_{nn}(z)$
	$S_{mn}(z) = H(z^{-1})S_{nn}(z)$
Properties	
$S_{nn}(z) = S_{nn}(z^{-1})$	
$T(z) = T(z^{-1})$	
$S_{nm}(z) = S_{mn}(z^{-1})$	

7-15 through 7-18 become:

$$S_{yy}(z) = [H(z)H(z^{-1})]S_{xx}(z) \quad (7\text{-}19)$$

where

$$T(z) = H(z)H(z^{-1})$$

and

$$S_{xy}(z) = S_{xx}(z)H(z) \quad (7\text{-}20)$$

Similarly,

$$S_{yx}(z) = S_{xx}(z)H(z^{-1}) \quad (7\text{-}21)$$

These results are tabulated in Figure 7-3(a). Before applying these formulas, we will comment on the symmetry properties of spectral functions.

7-4-1 Properties of Spectral Functions

The properties of spectral functions for continuous functions were developed in detail in Chapter 5. The proofs involving the spectral functions for discrete waveforms are almost identical to those for continuous waveforms except we use summations instead of integrals. Table 7-3 lists many of the main properties for discrete waveforms and a few of them will be demonstrated.

$S_{xx}(z)$, $S_{yy}(z)$, $T(z)$

Power spectral and power transfer functions have the same properties since they are the Z transforms of correlation functions.

EXAMPLE 7-11

Show that:

(a) $S_{xx}(z) = S_{xx}(z^{-1})$
(b) $S_{xy}(z) = S_{yx}(z^{-1})$

Solution

(a)
$$S_{xx}(z) = \sum_{-\infty}^{\infty} R_{xx}(n)z^{-n}$$

Let $p = -n$

and
$$S_{xx}(z) = \sum_{-\infty}^{\infty} R_{xx}(-p)z^{p}$$

$$= \sum_{-\infty}^{\infty} R_{xx}(p)z^{p},$$

(since $R_{xx}(p)$ is even)

Therefore
$$S_{xx}(z) = S_{xx}(z^{-1}) \quad (7\text{-}22)$$

Since $S_{xx}(z) = S_{xx}(z^{-1})$ we note that if $z - a$ is in the numerator or denominator, we must also have the term $(z^{-1} - a)$ or $(z - 1/a)$

present. Any power spectral density function or power transfer fu
tion

$$T(z) = Z[C_{hh}(n)]$$

has this property.

(b)

$$S_{xy}(z) = \sum_{-\infty}^{\infty} R_{xy}(n) z^{-n}$$

Let $p = -n$

Therefore

$$S_{xy}(z) = \sum_{-\infty}^{\infty} R_{xy}(-p) z^{p}$$

$$= \sum_{-\infty}^{\infty} R_{yx}(p) z^{p}$$

(since $R_{yx}(\tau) = R_{xy}(-\tau)$)

and

$$S_{xy}(z) = S_{yx}(z^{-1}) \quad (7$$

EXAMPLE 7-12

Given the pulse response of a system is:

$$h(n) = [(-0.6)^n + (0.5)^n]u(n)$$

use the Z transform to find the power transfer function and hence C_{hh}

Solution

$$h(n) = [(-0.6)^n + (0.5)^n]u(n)$$

therefore

$$H(z) = \frac{z}{z+0.6} + \frac{z}{z-0.5}, \qquad |z| > 0.6$$

$$= \frac{2z^2 + 0.1z}{(z+0.6)(z-0.5)}$$

$$T(z) = H(z)H(z^{-1})$$

$$= \frac{2z^2 + 0.1z}{(z+0.6)(z-0.5)} \frac{2z^{-2} + 0.1z^{-1}}{(z^{-1}+0.6)(z^{-1}-0.5)}$$

$$= \frac{2z^2 + 0.1z}{(z+0.6)(z-0.5)} \frac{2 + 0.1z}{(1+0.6z)(1-0.5z)}$$

$$= \frac{2z(z+0.05)0.1(z+20)}{0.6(z+0.6)(z+1.7)(z-0.5)(-0.5)(z-2)}$$

$$= \frac{-0.67z(z+0.05)(z+20)}{(z+0.6)(z+1.7)(z-0.5)(z-2)}, \qquad \text{for}$$

$$0.6 < |z| < 1.7$$

The correlation of $h(n)$ with itself may now be found using the inverse transform:

$$C_{hh}(n) = \frac{1}{2\pi j} \oint_C \frac{-0.67z(z+0.05)(z+20)}{(z+0.6)(z+1.7)(z-0.5)(z-2)} z^{n-1}\, dz$$

For $n \geq 0$

$$C_{hh}(n) = \sum [\text{residues of the poles at } z = -0.6 \text{ and } 0.5]$$

$$= \frac{-0.67(-0.55)(19.4)}{1.1(-1.1)(-2.6)}(-0.6)^n + \frac{-0.67(0.55)(20.5)}{1.1(2.2)-1.5)}(0.5)^n$$

$$= 2.27(-0.6)^n + 2.08(0.5)^n$$

For $n < 0$

We can now find $C_{hh}(n)$ as minus the residues of the poles at $z = -1.7$ and $z = 2$, or using the fact $R_{xx}(n) = R_{xx}(-n)$, we have:

$n < 0$

$$R_{xx}(n) = 2.27(-1.7)^n + 2.08(2)^n$$

$$= 2.27(-0.6)^{-n} + 2.08(0.5)^{-n}$$

7-4-2 Deterministic Signal Plus Uncorrelated Zero-Mean Noise

Table 7-3 shows a linear system with system function $H(z)$ and power transfer function $T(z) = H(z)H(z^{-1})$. The input is $x(k) = f(k) + n(k)$ where $f(k)$ is deterministic and $n(k)$ is a zero-mean uncorrelated noise waveform $[R_{fn}(k) = 0]$ with autocorrelation function $R_{nn}(k)$. In Chapter 3 we found the deterministic output as:

$$g(k) = f(k) * h(k) \tag{7-24}$$

and the output noise autocorrelation as:

$$R_{mm}(k) = C_{hh}(k) * R_{nn}(k)$$

and the cross-correlation of the input and noise as:

$$R_{nm}(k) = h(k) * R_{nn}(k)$$

and

$$R_{mn}(k) = R_{nm}(-k)$$

Using the Z transform, we obtain:

$$G(z) = F(z)H(z) \tag{7-25}$$

and as previously demonstrated:

$$S_{mm}(z) = Z(C_{hh}(n))S_{nn}(z)$$

$$= T(z)S_{nn}(z)$$

where $$T(z) = H(z)H(z^{-1})$$
$$S_{nm}(z) = H(z)S_{nn}(z)$$
and $$S_{mn}(z) = H(z^{-1})S_{nn}(z)$$

These results are summarized in Figure 7-3(b).

EXAMPLE 7-13

Consider a linear system with pulse response $h(n) = (0.6)^n u(n)$ has a input $x(k) = u(k) + n(k)$ where $n(k)$ is an ergodic noise waveform $R_{nn}(k) = 2\delta(n)$. Find the output signal for $k \gg 0$, the output noise po spectral density $S_{mm}(z)$, the output autocorrelation function, and the i and output signal to noise ratios for $k \gg 0$.

Solution

The Output Signal

$$H(z) = \frac{z}{z - 0.6}, \qquad |z| > 0.6$$

$$Y(z) = \frac{z}{z - 0.6}\frac{z}{z - 1}$$

$$\frac{Y(z)}{z} = \frac{z}{(z - 0.6)(z - 1)}$$

$$= \frac{-1.5}{z - 0.6} + \frac{2.5}{z - 1}$$

therefore $$Y(z) = \frac{-1.5z}{z - 0.6} + \frac{2.5z}{z - 1}$$

and $$y(n) = -1.5(0.6)^n u(n) + 2.5u(n)$$

and for $n \gg 0$, $y(n) = 2.5$.

The Output Noise Power Spectral Density

$$S_{mm}(z) = S_{nn}(z)T(z)$$

where $S_{nn}(z) = 2$ and $T(z) = H(z)H(z^{-1})$.

$$T(z) = \frac{z}{z - 0.6}\frac{z^{-1}}{z^{-1} - 0.6}$$

$$= \frac{z}{(z - 0.6)(-0.6)(z - 1.7)}$$

$$= \frac{-1.7z}{(z - 0.6)(z - 1.7)}$$

therefore $$S_{mm}(z) = \frac{-3.4z}{(z - 0.6)(z - 1.7)}, \qquad 0.6 \le |z| < 1.7$$

$$R_{mm}(n) = \frac{1}{2\pi j}\oint z^n \frac{-3.4}{(z - 0.6)(z - 1.7)}\,dz$$

For $n > 0$

$$R_{mm}(n) = [\text{residue of the pole at } z = 0.6]$$

$$= \frac{-3.4}{-1.1}(0.6)^n$$

$$= 3.1(0.6)^n$$

and by symmetry:

$$R_{mm}(n) = 3.1(0.6)^n u(n) + 3.1(1.7)^n u[-n - 1]$$

$$= 3.1(0.6)^{|n|}$$

Signal to Noise Ratios

At the input:

$$\frac{S}{N} = \frac{1}{R_{nn}(0)}$$

$$= 0.5$$

whereas at the output:

$$\frac{S}{N} = \frac{2.5^2}{3.1}$$

$$= 2.01$$

SUMMARY

The two-sided Z transform was defined as $F(z) = \Sigma_{n=-\infty}^{\infty} f(n)z^{-n}$ and if it exists it does so in an annulus $\rho_1 < |z| < \rho_2$. The behavior of $f(n)$ for $n < 0$ places the upper bound ρ_2 on $|z|$ and the behavior of $f(n)$ for $n > 0$ places the lower bound ρ_1 on $|z|$. The previously mastered material on the one-sided Z transform was utilized to facilitate the evaluation of two-sided transforms. If $f(n) = f_1(n)u(n) + f_2(n)u(-n)$, then $F(z) = F_1(z) + F_2(z)$, $\rho_1 < |z| < \rho_2$ where $F_1(z)$ is the one-sided Z transform of $f_1(n)$ and $F_2(z)$ is the one-sided Z transform of $f_2(-n)$ with z replaced by z^{-1}.

The most commonly occurring noncausal time functions are auto- and cross-correlation functions whether associated with ergodic noise waveforms or the correlation of the impulse response $h(n)$ with itself. In Chapter 3 $R_{xx}(n)$, $R_{xy}(n)$, $R_{yx}(n)$, and $C_{hh}(n)$ were defined and studied. Here their Z transforms, the spectral functions, $S_{xx}(z)$, $S_{xy}(z)$, $S_{yx}(z)$, and $T(z)$ were studied. Using famous transform pairs, we obtain:

$$Z[x(n)*y(n)] = X(z)Y(z)$$

$$Z[x(n) \oplus y(n)] = Y(z)X(z^{-1})$$

and

$$Z[y(n) \oplus x(n)] = X(z)Y(z^{-1})$$

The time-domain results for a linear discrete system with pulse resp $h(n)$ whose input is an ergodic noise waveform with autocorrelation fun $R_{xx}(n)$ were:

$$R_{yy}(n) = R_{xx}(n)*C_{hh}(n)$$

and

$$R_{xy}(n) = h(n)*R_{xx}(n)$$

Using the Z transform, we find:

$$S_{yy}(z) = S_{xx}(z)T(z)$$

and

$$S_{xy}(z) = H(z)S_{xx}(z)$$

$S_{yy}(z)$ and $S_{xx}(z)$ are power spectral densities, $S_{xy}(z)$ and $S_{yx}(z$ cross-spectral densities and $T(z) = Z[C_{hh}(n)] = H(z)H(z^{-1})$ is the p transfer function.

Inverse transforms were evaluated by either the use of partial fractio residues. If $F(z) = F_1(z) + F_2(z)$, where the poles of $F_1(z)$ are inside $|z| = \rho$ the poles of $F_2(z)$ are outside $|z| = \rho_2$, then the inverse is found as $f_1(n)$ $+ f_2(n)u(-n)$ by table reference.

Using residues the inverse transform $f(n)$ is defined as:

$$f(n) = \frac{1}{2\pi j}\oint_C z^{n-1}F(z)\,dz$$

and $f(n)$ is found as:

For $n \geq 0$

$$f(n) = \sum [\text{residues of the poles of } F(z)z^{n-1} \text{ inside } |z| = \rho]$$

For $n < 0$

$$f(n) = -\sum [\text{residues of the poles of } F(z)z^{n-1} \text{ outside } |z| = \rho]$$

PROBLEMS

7-1. Evaluate the two-sided Z transforms of the following functions:
(a) $f_1(n) = 3\delta(n+2) - \delta(n-1)$ **(b)** $f_2(n) = \sum_{k=0}^{\infty} a^{2k}\delta(n+2k)$
(c) $f_3(n) = 2$ **(d)** $f_4(n) = (3n-1)u(-n) + 3^n u(n)$
(e) $f_5(n) = (3n-1)u(-n-1) + 3^{n-1}u(n-1)$ **(f)** $f_6(n) = 2^n u(-n-1) + 3n(-1)^n u(n)$
(g) $f_7(n) = (3n^2 - 2n + 2)(0.5)^{-n}u(-n) + (3n-2)(0.5)^n u(n)$

(h) Without any work, what is the denominator polynomial and region of convergence of the Z transform of $(an^2 - b)(-2)^n u(n) + (cn + d)(0.7)^n u(-n-1)$?

7-2. Given:

$$x(n) \leftrightarrow X(z), \quad \rho_{11} < |z| < \rho_{12}$$
$$y(n) \leftrightarrow Y(z), \quad \rho_{21} < |z| < \rho_{22}$$
$$w(n) \leftrightarrow W(z), \quad \rho_{31} < |z| < \rho_{32}$$

Find the Z transform and its region of convergence for:
(a) $[x(n) \oplus y(n)]*z(n)$ **(b)** $[x(n)*y(n)] \oplus z(n)$
(c) $x(n) \oplus [y(n)*z(n)]$

7-3. **(a)** Prove whether or not:

$$[x(n) \oplus y(n)]*z(n) = x(n) \oplus [y(n)*z(n)]$$

(b) Under what conditions of evenness or oddness for $x(n)$ or $y(n)$ is:
(1) $x(n) \oplus y(n) = y(n) \oplus x(n)$
(2) $x(n) \oplus y(n) = x(n)*y(n)$
(3) $y(n) \oplus x(n) = x(n)*y(n)$

7-4. Evaluate the inverse transform of and plot $f(n)$ versus n for:
(a) $\frac{2z^3 + 3z^2 + 1}{z^2}$, for all z **(b)** $\frac{2z^2}{(z+1)^2}$, $|z| < 1$
(c) $\frac{2z^2}{(z+1)^2}$, $|z| > 1$ **(d)** $\frac{3z^3 + 2z^2 + z}{(z+3)^2(z-2)}$, $|z| < 2$
(e) $\frac{3z^3 + 2z^2 + z}{(z+3)^2(z-2)}$, $2 < |z| < 3$ **(f)** $\frac{3z^3 + 2z^2 + z}{(z+3)^2(z-2)}$, $|z| > 3$
(g) $\frac{z^6}{(z+3)^2(z-2)}$, $2 < |z| < 3$

7-5. If possible evaluate:
(a) $3^n u(-n) \oplus 2^n u(-n)$ **(b)** $3^n u(-n)*2^n u(-n)$
(c) $2(0.6)^{|n|} \oplus 2(0.6)^{|n|}$ **(d)** $2(0.6)^{|n|}*2(0.6)^{|n|}$
(e) $(3+n)(-0.5)^n u(n) \oplus (2+n)u(n)$ **(f)** $(3+n)(-0.5)^n u(n)*(2+n)u(n)$

7-6. Given a linear system with pulse response $h(n) = (-0.8)^n u(n)$ has as its input $x(n) = 4u(-n-1) + (0.6)^n u(n)$. Find the output $y(n)$.

7-7. If $x(n) \leftrightarrow X(z)$, $0.2 < |z| < 2$

and $y(n) \leftrightarrow Y(z)$, $0.8 < |z| < 3$

show:

$$Z[x(n)y(n)] = X(z)*Y(z) = \frac{1}{2\pi j}\oint_C X(p)Y\left(\frac{z}{p}\right)p^{-1}\,dp$$

Carefully explain for what annulus, $\rho_1 < |z| < \rho_2$, $X(z)*Y(z)$ exists and plot the poles and C on the p plane.

7-8. Use complex convolution to find the Z transform of $x(n)y(n)$, where:
(a) $x(n) = (-2)^n u(n)$ and $y(n) = nu(n)$
(b) $x(n) = 2^n u(-n-1) + u(n)$
$y(n) = (-0.6)^n u(-n-1) + (0.5)^n u(n)$

7-9. Given

$$\overline{x(n)} = X(z), \qquad \rho_{x1} < |z| = \rho < \rho_{x2}$$

and

$$\overline{y(n)} = Y(z), \qquad \rho_{y1} < |z| = \rho < \rho_{y2}$$

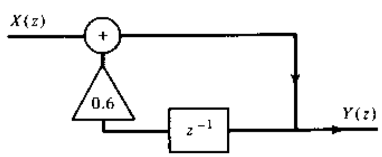
(a) What are the conditions for $x(n)$ and $y(n)$ to be stable?
(b) List when the following are stable, for $x(n)$ and $y(n)$ stable; and if they c unstable, give a specific example for $x(n)$ and $y(n)$:
(1) $x(n)y(n)$ (2) $x(n)*y(n)$ (3) $x(n) \oplus x(n)$
(4) $x(n) \oplus y(n)$ (5) $y(n) \oplus x(n)$
Sketch pole diagrams for each case.

7-10. Given a linear system with pulse response $h(n) = (-0.8)^n u(n)$ has as its i deterministic signal $f(n) = (-1)^n u(n)$ plus zero-mean independent white $n(k)$ with autocorrelation function $R_{nn}(k) = 6\delta(k)$:
(a) Find the output signal $g(n)$ and spectral densities $S_{mm}(z)$, $S_{nm}(z)$, and S
(b) Do the spectral functions possess their expected properties?
(c) Find the output mean square fluctuations $\overline{m^2(n)}$ using residue theory.

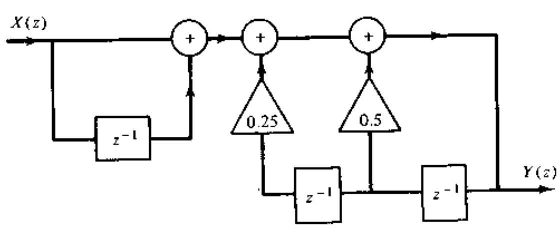
7-11. The power transfer function is defined as:

$$T(z) = H(z)H(z^{-1})$$

(a) Find the power transfer functions for the following systems and plot th zero diagram:
(1)



(2)



(b) If the input to each of the systems of part (a) is assumed white nois $S_{xx}(z) = 2$, use residue theory to find the mean-squared output fluctu

$$\overline{y^2(n)} = R_{yy}(0) = \frac{1}{2\pi j}\oint S_{yy}(z)z^{-1}\,dz$$

at the output.

7-12. (a). If the input to system (1) of the previous problem has $R_{xx}(n) = 10(0.6)^{|n|}$, find the power spectral density of the output noise and the cross-spectral density of the input and output noise $S_{xy}(z)$.
(b). Give a pole zero plot for $S_{xy}(z)$, $S_{yx}(z)$, and $S_{yy}(z)$.
(c). Find the signal to noise ratio at the input and output for $n \gg 0$ if an input signal $f(n) = 6u(n)$ is added to the input noise.

7-13. A power spectral density $S_{xx}(z)$ or power transfer function $T(z)$ may be written as:

$$S_{xx}(z) = G(z)G(z^{-1}) \quad \text{or} \quad T(z) = H(z)H(z^{-1})$$

where $G(z)$ and $H(z)$ have their poles or zeros inside the unit circle $z = 1$.

Design a "shaping filter" that transforms white noise with $S_{xx}(z) = 1$ to noise with a power spectral density:

$$S_{yy}(z) = \frac{-1.5z}{(z-0.5)(z-2)}, \qquad 0.5 < |z| < 2$$

7-14. Which of the following functions qualify as power spectral densities?

(a) $\dfrac{-0.5z}{z^2 + 2.5z + 1}$ (b) $\dfrac{0.5z}{z^2 + 2.5z + 1}$

(c) $\dfrac{0.5}{z^2 + 2.5z + 1}$ (d) $\dfrac{-z}{z^2 - 16}$

(e) $\dfrac{2z}{(z+2)^2(z+0.5)^2}$