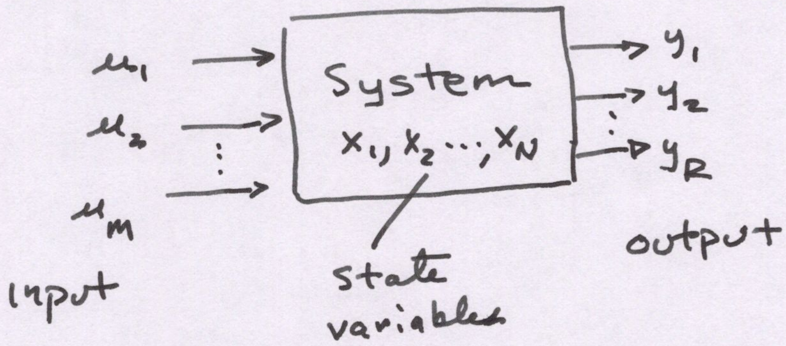
Modern Control

State Equations



input vector $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_M \end{bmatrix}$

output vector

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_R \end{bmatrix}$$

state vector $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$

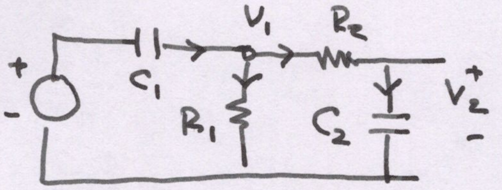
Any LTI system can be expressed in matrix form as

$$\dot{\underline{x}} = [A]\underline{x} + [B]\underline{u}$$

$$\underline{y} = [C]\underline{x} + \cancel{[D]\underline{u}} \to 0$$

by assumption in this course

Consider

input  output is V_2

using KCL

$$\frac{dv_1}{dt} = v_1\left\{-\frac{1}{R_1C_1} - \frac{1}{R_2C_1}\right\} + v_2\left\{\frac{1}{C_1R_2}\right\} + \frac{dv_{in}}{dt}$$

$$\frac{dv_2}{dt} = v_1\left\{+\frac{1}{C_2R_2}\right\} + v_2\left\{-\frac{1}{C_2R_2}\right\}$$

state variable $v_1, v_2 \Rightarrow \begin{array}{l} x_1 \equiv v_1 \\ x_2 \equiv v_2 \end{array}$

input $\frac{dv_{in}}{dt} \Rightarrow u_1$

output $V_2 \Rightarrow y = V_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1C_1} - \frac{1}{R_2C_1} & \frac{1}{C_1R_2} \\ \frac{1}{C_2R_2} & -\frac{1}{R_2C_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Consider

$$\frac{Y}{U} = \frac{20\,(s+2)}{s^2+4s+14}$$

Phasevariable representation

$$\frac{Y}{U} = \frac{X}{U}\,\frac{Y}{X}$$

if we define

$$\frac{Y}{X} = (s+2) \Rightarrow y = \dot{x} + 2x$$

$$\frac{X}{U} = \frac{20}{s^2+4s+14} \Rightarrow \ddot{x} + 4\dot{x} + 14x = u$$

define

$$x_1 \equiv x$$

$$x_2 \equiv \dot{x}$$

then $\boxed{\dot{x}_1 = x_2}$

and $\ddot{x} + 4\dot{x} + 14x = u \Rightarrow \boxed{\dot{x}_2 + 4x_2 + 14x_1 = u}$

$$y = \dot{x} + 2x \rightarrow \boxed{y = x_2 + 2x_1}$$

$$\boxed{\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -14 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}}$$

Once we have the matrices A, B, C how do we solve for $\underline{x}$?

Homogeneous soln

given $\dot{\underline{x}} = A\underline{x} + B\underline{u}$

let $\underline{u} = 0$

$$\dot{\underline{x}} = A\underline{x}$$

say $\underline{x} = e^{\lambda t}\underline{c} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix} = e^{\lambda t}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}$

then $\dot{\underline{x}} = \lambda e^{\lambda t}\underline{c} = \lambda e^{\lambda t}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}$

so $\boxed{A\underline{c} = \lambda\underline{c}}$ where $\underline{c}$ is unknown ~~of~~ and λ is unkown

The problem is term

$$\left[A - \lambda I\right]\underline{c} = \underline{0}$$

the eigenvalue (λ) eigenvector ($\underline{c}$) problem

solved numerially

Why? one cannot analytically solve the characteristic equation for degree higher than 3.

Look at a simple problem

$$\dot{\underline{x}} = \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}_{A} \underline{x}$$

$$[A - I\lambda] = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$[A - I\lambda] = \begin{bmatrix} 2-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}$$

char. eqn: $\begin{vmatrix} 2-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 0(1) = 0$

$$\left.\begin{array}{l} \lambda_1 = 2 \\ \lambda_2 = 1 \end{array}\right\} \text{eigenvalues}$$

Determine eigenvector for each eigenvalue

$\lambda_1 = 2$

$$\begin{bmatrix} 2-\lambda_1 & 1 \\ 0 & 1-\lambda_1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} c_2 = 0 \\ c_1 = \text{any value} \end{array}$$

$$\therefore \underline{c}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\lambda_2 = 1$

$$\begin{bmatrix} 2-\lambda_2 & 1 \\ 0 & 1-\lambda_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} c_1 + c_2 = 0 \\ \therefore c_2 = -c_1 \end{array}$$

however c_1 = any value

$$\underline{c}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Summary of the result

$$\underline{c}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \lambda_1 = 2$$

$$\underline{c}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \lambda_2 = 1$$

The solution

$$\underline{x} = B_1 \underline{c}_1 e^{\lambda_1 t} + B_2 \underline{c}_2 e^{\lambda_2 t}$$

$$\therefore \quad \boxed{\begin{aligned} x_1 &= B_1 (0) e^{2t} + B_2 (1) e^{1.t} \\ x_2 &= B_1 (1) e^{2t} + B_2 (-1) e^{1.t} \end{aligned}}$$

Back to the general problem

if A has unique eigenvalues

$$\lambda's = \lambda_1, \lambda_2, \lambda_2 \dots \lambda_N \qquad \lambda_i \neq \lambda_j \text{ when } i \neq j$$

$$\underline{e}'s \Rightarrow \underline{e}_1, \underline{e}_2 \dots \quad \underline{e}_N$$

Then

$$\left.\begin{aligned} A\underline{e}_1 &= \lambda_1 \underline{e}_1 \\ A\underline{e}_2 &= \lambda_2 \underline{e}_2 \\ &\vdots \\ A\underline{e}_N &= \lambda_N \underline{e}_N \end{aligned}\right\} A\left[\underline{e}_1 \,\vdots\, \underline{e}_2 \dots \middle| \underline{e}_N\right] = \left[\lambda_1\underline{e}_1 \,\vdots\, \lambda_2 \underline{e}_2 \dots \lambda_N \underline{e}_N\right]$$

$$\underbrace{A}_{A}\underbrace{\left[\underline{e}_1 \middle| \underline{e}_2 \dots \underline{e}_N\right]}_{E} = \underbrace{\left[\underline{e}_1 \middle| \underline{e}_2 \dots \underline{e}_N\right]}_{E}\underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix}}_{\Lambda}$$

$$A E = E \Lambda$$

$$\boxed{A = E \Lambda E^{-1}}$$

where E is comprised of the eigenvectors of A

The solution of $\dot{\underline{x}} = A\underline{x}$ given A has unique eigenvalues

Step 1 let $\underline{x} = E\underline{z}$ than $\dot{\underline{x}} = E\dot{\underline{z}}$

subs. into $\dot{\underline{x}} = A\underline{x}$ yields

$$\boxed{E\dot{\underline{z}} = AE\underline{z}}$$

Step 2 Premult by E^{-1}

$$E^{-1}E\dot{\underline{z}} = E^{-1}AE\underline{z}$$

$$\boxed{\dot{\underline{z}} = E^{-1}AE\underline{z}}$$

Step 3 Since $A = E\Lambda E^{-1}$

then $\boxed{E^{-1}AE = \Lambda}$

proof: $E^{-1}A = E^{-1}(E\Lambda E^{-1})$

$E^{-1}AE = E^{-1}(E\Lambda E^{-1})E$

$\quad " \quad = \Lambda$

$$\boxed{\dot{\underline{z}} = \Lambda\underline{z}}$$

where $\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix}$

soln for $\underline{z}(t)$

$$\boxed{\underline{z} = \left[e^{\Lambda t}\right]\underline{z}(0) \quad \text{where } e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_N t} \end{bmatrix}}$$

Proof by example that $\underline{z}(t) = e^{\Lambda t}\underline{z}(0)$

consider

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\dot{z}_1 = \lambda_1 z_1 \Rightarrow z_1(t) = e^{\lambda_1 t} z_1(0)$$

$$\dot{z}_2 = \lambda_2 z_2 \Rightarrow z_2(t) = e^{\lambda_2 t} z_2(0)$$

in matrix form

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix}$$

Step 4 Transform back to $\underline{x}(t)$

$$\underline{x}(t) = E\,\underline{z}(t)$$

$$\Downarrow$$

$$E^{-1}\underline{x}(t) = \underline{z}(t)$$

then $\underline{z}(t) = e^{\Lambda t}\underline{z}(0)$

$$\Downarrow$$

$$E^{-1}\underline{x}(t) = e^{\Lambda t} E^{-1}\underline{x}(0)$$

$$\Downarrow$$

$$\boxed{\underline{x}(t) = \left(E e^{\Lambda t} E^{-1}\right)\underline{x}(0)}$$

<u>Finally</u> $\underline{\dot{x}} = A\underline{x}$

then soln

$$\underline{x} = \underline{\Phi}\, \underline{x}(0)$$

where $\underline{\Phi} = E e^{\Lambda t} E^{-1}$ "State transition ~~s~~matrix"

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example

given $\underline{\dot{x}} = A\underline{x}$

then eigenvectors and value ~~are~~ of

A are computed at

$$\lambda_1 = 2;\ \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = -1;\ \underline{e}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Determine $\underline{\Phi}$

$$E = \left[\begin{array}{c:c} 1 & 1 \\ 0 & 2 \end{array}\right] = \left[\underline{e}_1 \,\vdots\, \underline{e}_2\right]$$

$$E^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{Bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{Bmatrix}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix}$$

$$\underline{\Phi} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{-1}$$

$$\underline{\Phi} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & -1/2 \\ 1 & 1/2 \end{bmatrix}$$

$$\boxed{\underline{\Phi} = \begin{bmatrix} e^{2t} & \dfrac{e^{-t} - e^{2t}}{2} \\ 0 & e^{-t} \end{bmatrix}}$$

Using maxima to compute the STM

```
E: matrix([1,1],[0,2]);

EINV: E^^(-1);

EL : matrix([%e^(2*t),0],[0,%e^(-1*t)]);

STM : E. EL.EINV;

quit();
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