may be written as:

$$f(t) = f_T(t) * \sum_{n=0}^{\infty} \delta(t - nT)$$

(c) Using the convolution theorem, verify the Laplace transform of the semi periodic function $f(t)$ is:

$$F(s) = \frac{F_T(s)}{1 + e^{-Ts}}$$

(d) Find the inverse transform of:

$$G(s) = \frac{1 - e^{-s}}{s(1 - e^{-4s})}, \qquad \text{Re}(s) > 0$$

and plot $g(t)$.

(e) Plot the pole-zero diagram of $G(s)$ and verify your result finding $g(t)$ by using the residue theory.

4-10. Given:

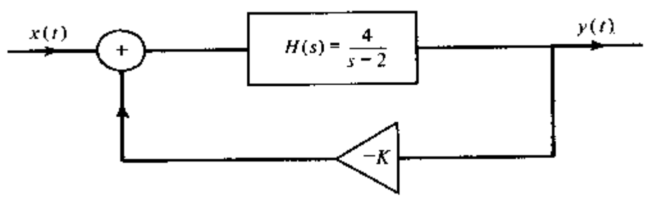
$$g(t) = \sum_{n=0}^{\infty} f(t - 2n)$$

where

$$f(t) = 1, \qquad 0 < t < 1$$
$$= 0, \qquad \text{otherwise}$$

is the input to a system with system function $H(s) = 2/(s + 3)$. Find and plot the zero-state output $y(t)$.

4-11. (a) Find $\theta(s) = Y(s)/X(s)$ for the system shown.
(b) For what values of K is the system stable?



Chapter 5*

The Two-sided Laplace Transform

INTRODUCTION

Chapter 5 treats the two-sided or bilateral Laplace transform whose main application is the solving of LTIC continuous systems with random or signal plus random inputs. The chapter considers the transform analysis of the continuous material of Chapter 3.

The material is traversed by means of what is now our standard treatment of any transform. The different stages are:

1. The transform is defined and a number of transforms are evaluated. We utilize to the fullest our knowledge of one-sided transforms to evaluate two-sided ones.
2. The properties and theorems are given and attention is focused on the transform of convolution and correlation integrals.
3. The inverse transform is treated either by using previously mastered partial fraction techniques referring to tables, or using the residue theory from complex variables.
4. LTIC systems are solved for the cases of random and signal plus noise inputs.

5-1 DEFINITION AND EVALUATION OF SOME TRANSFORMS

The **two-sided** or **bilateral Laplace transform** of a real function $f(t)$ is defined as:

$$F_B(s) \triangleq \int_{-\infty}^{\infty} f(t)e^{-st}\,dt \qquad (5\text{-}1)$$

for complex $s = \sigma + j\omega$. $F_B(s)$ if it exists will do so for a region of the complex plane, $\sigma_1 < \text{Re}(s) < \sigma_2$, called the **region of convergence.** Normally, the subscrip "B" is excluded and it will be clear from the context whether the one- o two-sided Laplace transform is being used. Again, alternate notations, $\mathcal{L}[f(t)]$ $\overline{f(t)}$ and $f(t) \leftrightarrow F(s)$ will be used.

A number of transforms will now be evaluated and the concept of th region of convergence will be expanded.

EXAMPLE 5-1

Find the two-sided Laplace transforms of the following functions and stat the region of convergence:

(a) $f_1(t) = 3e^{2t}u(t)$
(b) $f_2(t) = 3e^{2t}u(-t)$
(c) $f_3(t) = 3e^{-2t}u(t) + 4e^{t}u(-t)$
(d) $f_4(t) = 3e^{-2t}u(t) - 4e^{t}u(t)$
(e) $f_5(t) = -3e^{-2t}u(-t) + 4e^{t}u(-t)$
(f) $f_6(t) = A_1e^{\alpha t}u(t) + A_2e^{\beta t}u(-t)$ in general for all appropriate real α an β.

Solution

(a) $f_1(t) = 3e^{2t}u(t)$

Therefore
$$F_1(s) = \int_{-\infty}^{0} 0e^{-st}\,dt + \int_0^{\infty} 3e^{2t}e^{-st}\,dt$$
$$= \frac{3}{s-2}, \qquad \text{Re}(s) > 2$$

$f_1(t)$, $F_1(s)$, and the region of convergence are shown in Figure 5-1(a).

(b) $f_2(t) = 3e^{2t}u(-t)$

$$F_2(s) = \int_{-\infty}^{0} 3e^{2t}e^{-st}\,dt$$
$$= \int_{-\infty}^{0} 3e^{(2-s)t}\,dt$$
$$= \frac{3}{2-s}e^{(2-\sigma)t}e^{-j\omega t}\bigg|_{-\infty}^{0}, \qquad \text{writing } s = \sigma + j\omega$$
$$= \frac{3}{2-s} - 0, \qquad \text{if } \sigma < 2$$

since then $e^{(2-\sigma)(-\infty)} = 0$ for $2 - \sigma > 0$

Therefore
$$F_2(s) = -\frac{3}{s-2}, \qquad \text{Re}(s) < 2$$

$f_2(t)$, $F_2(s)$, and the region of convergence are shown in Figure 5-1(b).

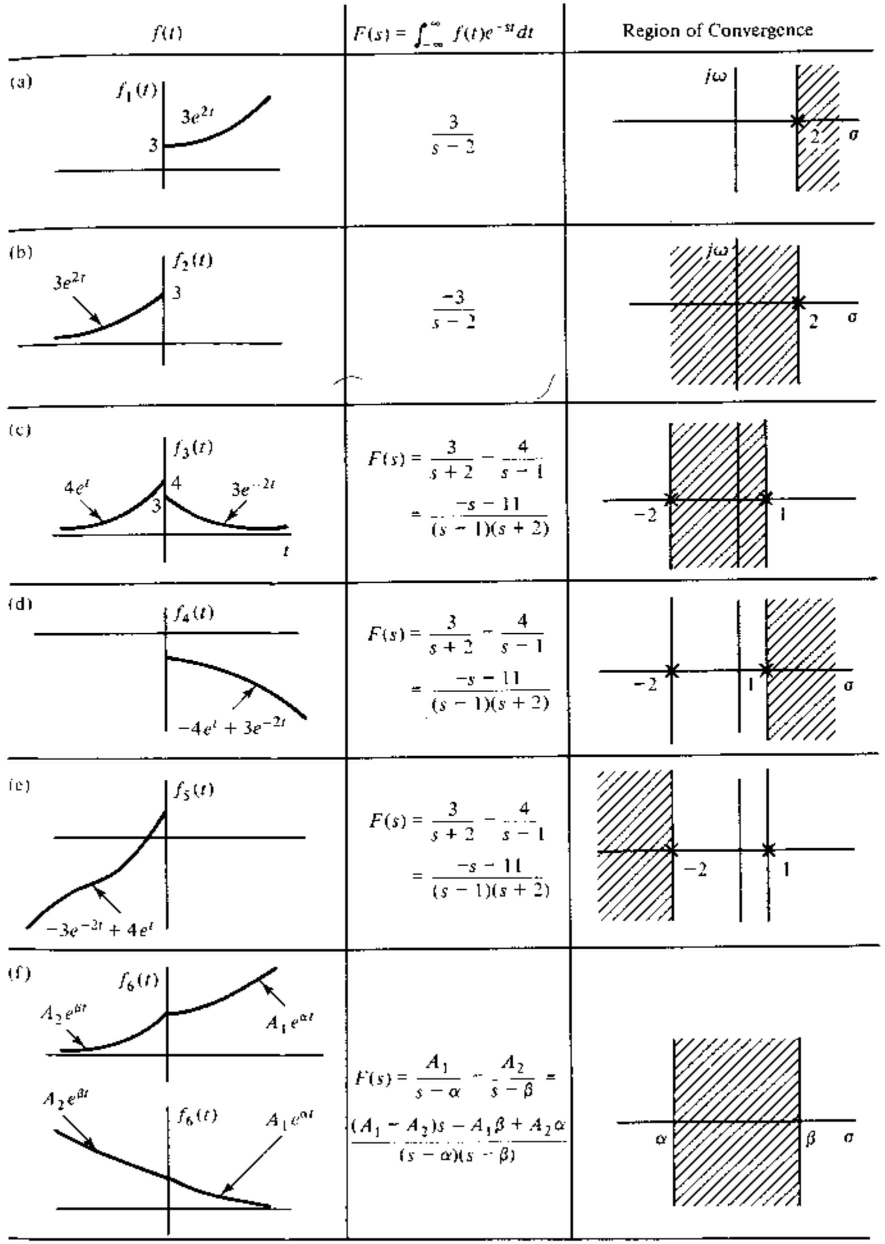


Figure 5-1 The two-sided Laplace transforms and their regions of convergence for Example 5-1.

(c) $f_3(t) = 3e^{-2t}u(t) + 4e^{+t}u(-t)$

$$F_3(s) = \int_{-\infty}^{0} 4e^{t}e^{-st}\,dt + \int_{0}^{\infty} 3e^{-st}e^{-2t}\,dt$$

$$= \frac{4}{1-s}e^{(-1-s)t}\Big|_{-\infty}^{0} + \frac{3}{-s-2}e^{(-2-s)t}\Big|_{0}^{\infty}$$

$$= \frac{-4}{s-1} + 0\Big|_{\text{if } \sigma<1} + 0\Big|_{\text{if } \sigma>-2} + \frac{3}{s+2}$$

$$= \frac{-s-11}{(s-1)(s+2)}; \qquad -2 < \mathrm{Re}(s) < +1$$

For this function which exists for both positive and negative time we note that the behavior for negative time puts an upper bound on an allowable σ and the function behavior for positive time puts a lower bound on an allowable σ. Therefore we obtain the strip of convergence $\sigma_1 < \sigma < \sigma_2$. $f_3(t)$, $F_3(s)$, and the region of convergence are plotted in Figure 5-1(c).

(d) $f_4(t) = (3e^{-2t} - 4e^{+t})u(t)$

$$F_4(s) = \frac{3}{s+2} + 0\Big|_{\text{if } \sigma>-2} - \frac{4}{s-1} + 0\Big|_{\text{if } \sigma>+1}$$

$$= \frac{-s-11}{(s-1)(s+2)}, \qquad \mathrm{Re}(s) > 1$$

We notice the bilateral transform expression is the same as in part (c) but the region of convergence is different. $f_4(t)$, $F_4(s)$, and the region of convergence are shown in Figure 5-1(d).

(e) $f_5(t) = -3e^{-2t}u(-t) + 4e^{t}u(-t)$

$$F_5(s) = \frac{3}{s+2} + 0\Big|_{\text{if } \sigma<-2} - \frac{4}{s-1} + 0\Big|_{\text{if } \sigma<1}$$

Therefore $$F_5(s) = \frac{-s-11}{(s-1)(s+2)}, \qquad \mathrm{Re}(s) < -2$$

Since $e^{-2t}u(-t)$ is an increasing exponential of negative time its convergence factor $\sigma < -2$ determines the region of convergence. Again, the Laplace transform expression is identical to parts (c) and (d) but the region of convergence is different. $f_5(t)$, $F_5(s)$, and the region of convergence are plotted in Figure 5-1(e).

(f) $f_6(t) = A_1e^{\alpha t}u(t) + A_2e^{\beta t}u(-t)$

It can quite easily be shown or by now be clear that:

$$F_6(s) = \frac{A_1}{s-\alpha} + 0\Big|_{\text{if } \sigma>\alpha} - \frac{A_2}{s-\beta} + 0\Big|_{\text{if } \sigma<\beta}$$

$$= \frac{(A_1-A_2)s - A_1\beta + A_2\alpha}{(s-\alpha)(s-\beta)} \qquad \text{if } \sigma > \alpha \cap \sigma < \beta$$

The bilateral Laplace transform will exist for all α and β such that $\alpha < \beta$ and then the region of convergence is $\alpha < \mathrm{Re}(s) < \beta$. The different possible situations for $f_6(t)$ are shown in Figure 5-1(f).

5-1-1 Two-sided Transforms Using One-sided Transforms

The evaluation of two-sided Laplace transforms involves the same amount of work as doing two one-sided Laplace transforms and indeed a table of one-sided Laplace transforms may be used to find two-sided ones. We will now develop this technique. Given:

$$f(t) = f_1(t)u(t) + f_2(t)u(-t)$$

$$F(s) = F_1(s) + \int_{-\infty}^{0} f_2(t)e^{-st}\,dt$$

Letting $t = -p$, we obtain:

$$dt = -dp, \qquad -\infty < t < 0 \text{ gives } \infty > p > 0$$

$$F(s) = F_1(s) + \int_{\infty}^{0} f_2(-p)e^{ps}\,(-dp)$$

and $$F(s) = F_1(s) + \mathcal{L}[f_2(-t)u(t)]\Big|_{s=-s} \tag{5-2}$$

If $f_2(-t)u(t)$ has a region of convergence $\sigma > \sigma_2'$, then $f_2(t)u(-t)$ has a region of convergence $\sigma < -\sigma_2'$.

EXAMPLE 5-2

Using Equation 5-2, find the two-sided Laplace transform of:

(a) $f(t) = e^{2t}u(-t)$
(b) $f(t) = 3e^{-2t}u(t) + te^{-t}u(-t)$

Solution

(a) $F(s) = \mathcal{L}[e^{-2t}u(t)]\Big|_{s=-s}$

$$= \frac{1}{s+2}\Big|_{s=-s} = \frac{-1}{s-2}, \qquad \sigma < 2$$

(b) $f(t) = 3e^{-2t}u(t) + te^{-t}u(-t)$

Using Equation 5-2, we obtain:

$$F(s) = \frac{3}{s+2} + \mathcal{L}[-te^{t}u(t)]\Big|_{s=-s}$$

$$= \frac{3}{s+2} + \frac{-1}{(s-1)^2}\Big|_{s=-s}, \qquad \sigma > -2 \cap \sigma < -1$$

$$= \frac{3}{s+2} - \frac{1}{(1+s)^2}$$

$$= \frac{3s^2 + 5s + 1}{(s+2)(s-1)^2}, \qquad -2 < \sigma < -1$$

Example 5-1 should now be repeated using Equation 5-1.

There are a few very important functions for which a two-sided Laplace transform does not exist. The functions $f_1(t) = 1, f_2(t) = \text{Cos}\,(\omega_0 t + \phi)$, and $f(t) = \Sigma_{n=-\infty}^{\infty} g(t - nT)$ a periodic function, do not possess a Laplace transform as each function for $t < 0$ requires $\sigma < 0$, whereas the function behavior for $t > 0$ requires $\sigma > 0$. Therefore, no value of s exists for which the transform converges. These functions are not to be confused with the causal functions:

$$f_1(t) = u(t), \quad f_2(t) = \text{Cos}\,(\omega_0 t + \phi)u(t), \quad \text{and} \quad f_3(t) = \left[\sum_{n=-\infty}^{n=\infty} g(t - nT)\right]u(t)$$

Finally, we are rarely interested in the two-sided Laplace transforms of functions whose transforms are not the ratio of two polynomials in s. As we will see, the most common functions for which we find two-sided Laplace transforms are correlation functions for finite energy waveforms and autocorrelation functions for ergodic noise waveforms. Both these type functions are *even* and for positive or negative time may often be represented by the product of polynomials and exponentials.

5-2 IMPORTANT THEOREMS OF BILATERAL LAPLACE TRANSFORMS

Table 4-2 of Chapter 4 listed an extensive set of theorems and properties of the one-sided Laplace transform. Some important theorems for the bilateral Laplace transform are given in Table 5-1. Since the convolution and correlation theorems are of the utmost importance when applying this material to systems with random inputs we will prove and demonstrate these. Discussion of the complex convolution theorem is deferred until after the inverse transform is considered.

EXAMPLE 5-3

Prove the convolution theorem and comment on the region of convergence.

Solution. The convolution theorem states that if:

$$f(t) \leftrightarrow F(s), \qquad \sigma_{f_1} < \sigma < \sigma_{f_2}$$

and

$$g(t) \leftrightarrow G(s), \qquad \sigma_{g_1} < \sigma < \sigma_{g_2}$$

then

$$f(t) * g(t) \leftrightarrow F(s)G(s)$$

TABLE 5-1

Theorem	Time function	Two-sided Laplace transform
Linearity	$ax(t) + by(t)$	$aX(s) + bY(s)$ $\sigma > \max(\sigma_{x1}, \sigma_{y1}), \sigma < \min(\sigma_{x2}, \sigma_{y2})$
Time-scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$, $a\sigma_1 < \sigma < a\sigma_2, \; a > 0$; $a\sigma_2 < \sigma < a\sigma_1, \; a < 0$
Shifting	$x(t - a)$	$e^{-as}X(s), \quad \sigma_1 < \sigma < \sigma_2$
Convolution	$x(t)*y(t)$ $= \int_{-\infty}^{\infty} x(p)y(t-p)\,dp$	$X(s)Y(s), \quad (\sigma_{x_1} < \sigma < \sigma_{x_2}) \cap (\sigma_{y_1} < \sigma < \sigma_{y_2})^a$
Cross-correlation	$x(t) \oplus y(t)$ $= \int_{-\infty}^{\infty} y(p)x(p-t)\,dp$	$Y(s)X(-s), \quad (\sigma_{y_1} < \sigma < \sigma_{y_2}) \cap (-\sigma_{x_2} < \sigma < -\sigma_{x_1})^a$
Autocorrelation	$x(t) \oplus x(t)$ $= \int_{-\infty}^{\infty} x(p)x(p-t)\,dp$	$X(s)X(-s), \quad \max(\sigma_1, -\sigma_2) < \sigma < \min(\sigma_2, -\sigma_1)^a$

For causal functions all one-sided Laplace transform theorems carry over.

[a] $x(t) \leftrightarrow X(s), \quad \sigma_1 < \sigma < \sigma_2$ or $\sigma_{x_1} < \sigma < \sigma_{x_2}$
$y(t) = Y(s), \quad \sigma_{y_1} < \sigma < \sigma_{y_2}$

By definition:

$$\mathcal{L}[f(t)*g(t)] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(p)g(t-p)\,dp\right]e^{-st}\,dt$$

Interchanging the order of integration, we obtain:

$$\mathcal{L}[f(t)*g(t)] = \int_{-\infty}^{\infty} f(p)\left[\int_{-\infty}^{\infty} g(t-p)e^{-st}\,dt\right]dp$$

Letting $t - p = u$:

$$dt = du, \qquad -\infty < t < \infty \qquad \text{gives} \qquad -\infty < u < \infty$$

$$\mathcal{L}[f(t)*g(t)] = \int_{-\infty}^{\infty} f(p)\left[\int_{-\infty}^{\infty} g(u)e^{-s(u+p)}\,du\right]dp$$

$$= \int_{-\infty}^{\infty} f(p)e^{-sp}G(s)\,dp$$

$$= G(s)F(s) \tag{5-3}$$

If we write $R(s) = G(s)F(s)$, then $R(s)$ will converge for all s for which both $G(s)$ and $F(s)$ converge. For example, if $F(s)$ converges $-3 < \sigma < 1$ and $G(s)$ converges $-2 < \sigma < 4$, then $R(s)$ would converge $-2 < \sigma < 1$. In general, $R(s)$ converges over the intersection of the points

$$(\sigma_{f_1} < \sigma < \sigma_{f_2}) \cap (\sigma_{g_1} < \sigma < \sigma_{g_2}) \tag{5-4}$$

EXAMPLE 5-4

Find the transforms of the following functions and denote the regions of convergence if the transform exists:

(a) $f_1(t) = e^{-2t}u(t) * e^{t}u(-t)$
(b) $f_2(t) = e^{2t}u(t) * e^{-t}u(-t)$
(c) $e^{-2|t|} * u(t)$

Solution

(a) Let $x(t) = e^{-2t}u(t)$

and then $$X(s) = \frac{1}{s+2}, \qquad \sigma > -2$$

Let $y(t) = e^{t}u(-t)$

and then $$Y(s) = -\frac{1}{s-1}, \qquad \sigma < 1$$

By the convolution theorem:

$$\mathcal{L}[x(t)*y(t)] = \frac{-1}{(s+2)(s-1)}, \qquad \sigma > -2 \cap \sigma < +1$$
$$= \frac{-1}{(s+2)(s-1)}, \qquad -2 < \sigma < +1$$

(b) Let $x(t) = e^{2t}u(t)$

and then $$X(s) = \frac{1}{s-2}, \qquad \sigma > 2$$

Let $y(t) = e^{-t}u(-t)$

and then $$Y(s) = \frac{-1}{s+1}, \qquad \sigma < -1$$

By the convolution theorem:

$$\mathcal{L}[x(t)*y(t)] = \frac{-1}{(s-2)(s+1)}, \qquad \sigma > 2 \cap \sigma < -1$$

Since $\sigma > 2 \cap \sigma < -1 = \phi$ the Laplace transform does not exist.

(c) Let $x(t) = e^{-2|t|} = e^{-2t}u(t) + e^{2t}u(-t)$ and this has a Laplace transform:

$$X(s) = \frac{1}{s+2} - \frac{1}{s-2} = \frac{-4}{(s+2)(s-2)}, \qquad -2 < \sigma < 2$$

Let $y(t) = u(t)$

and then $$Y(s) = \frac{1}{s}, \qquad \sigma > 0$$

By the convolution theorem:

$$\mathcal{L}[x(t)*y(t)] = \frac{-4}{s(s+2)(s-2)}, \qquad -2 < \sigma < 2 \cap \sigma > 0$$
$$= \frac{-4}{s(s+2)(s-2)}, \qquad 0 < \sigma < 2$$

Our anticipation of evaluating inverse two-sided transforms should be mounting since we now have an alternative to convolution.

EXAMPLE 5-5

Prove the correlation theorems:

(a) $\mathcal{L}[x(t) \oplus y(t)] = Y(s)X(-s)$
(b) $\mathcal{L}[x(t) \oplus x(t)] = X(s)X(-s)$

Solution

(a) By definition:

$$\mathcal{L}[x(t) \oplus y(t)] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} y(p)x(p-t)\,dp\right]e^{-st}\,dt$$

Interchanging the order of integration, we obtain:

$$\mathcal{L}[x(t) \oplus y(t)] = \int_{-\infty}^{\infty} y(p)\left[\int_{-\infty}^{\infty} x(p-t)\,e^{-st}\,dt\right]dp$$

Substitute $p - t = l$

$$\mathcal{L}[x(t) \oplus y(t)] = \int_{-\infty}^{\infty} y(p)\left[\int_{+\infty}^{-\infty} x(l)e^{-s(p-l)}\,(-dl)\right]dp$$
$$= \int_{-\infty}^{\infty} y(p)e^{-sp}\,dp \int_{-\infty}^{\infty} x(l)e^{sl}\,dl = Y(s)X(-s) \qquad (5\text{-}5)$$

where $C(s) = Y(s)X(-s)$ converges for all s for which both $Y(s)$ and $X(-s)$ converge. To interprete the region of convergence for $X(-s)$, we note that if $X(s)$ contains a pole at $s = -s_1$, then $X(-s)$ contains a pole at $s = s_1$. With a little more thought we conclude that if the region of convergence for $X(s)$ is $\sigma_{x_1} < \sigma < \sigma_{x_2}$, then the region of convergence for $X(-s)$ is $\sigma_{x_1} < -\sigma < \sigma_{x_2}$ or $-\sigma_{x_2} < \sigma < -\sigma_{x_1}$ and the region of convergence for $C(s)$ is the intersection of the points:

$$(\sigma_{y_1} < \sigma < \sigma_{y_2}) \cap (-\sigma_{x_2} < \sigma < -\sigma_{x_1}) \qquad (5\text{-}6)$$

(b) In the case $y(t) = x(t)$ we find by the cross-correlation theorem that:

$$\mathcal{L}[x(t) \oplus x(t)] = X(s)X(-s) \qquad (5\text{-}7)$$

If the region of convergence for $X(s)$ is $\sigma_1 < \sigma < \sigma_2$, then $X(s)X(-s)$ has a region of convergence $\max(\sigma_1, -\sigma_2) < \sigma < \min(\sigma_2, -\sigma_1)$ if it exists.

EXAMPLE 5-6

Find the Laplace transforms of the following correlation functions:

(a) $e^{-2t}u(t) \oplus e^{-2t}u(t) = C_{xx}(t)$
(b) $e^{-t}u(t) \oplus e^{2t}u(-t) = C_{xy}(t)$
(c) $e^{2t}u(-t) \oplus e^{-t}u(t) = C_{yx}(t)$

Solution

(a) $e^{-2t}u(t) \leftrightarrow \dfrac{1}{s+2}$

$$\mathcal{L}[e^{-2t}u(t) \oplus e^{-2t}u(t)] = \frac{1}{(s+2)}\frac{1}{(-s+2)}$$

$$= \frac{-1}{(s+2)(s-2)}, \qquad \sigma > -2 \cap \sigma < 2$$

$$= \frac{-1}{s^2-4}, \qquad -2 < \sigma < +2$$

(b) $x(t) = e^{-t}u(t) \leftrightarrow \dfrac{1}{s+1}, \qquad \sigma > -1,$

$$y(t) = e^{2t}u(-t) \leftrightarrow \frac{-1}{s-2}, \qquad \sigma < -2$$

Therefore $\quad \mathcal{L}[e^{-t}u(t) \oplus e^{2t}u(-t)]$

$$= \frac{-1}{s-2}\frac{1}{-s+1}$$

$$= \frac{1}{(s-2)(s-1)}, \qquad \sigma < 2 \cap \sigma < 1$$

$$= \frac{1}{(s-2)(s-1)}, \qquad \sigma < 1$$

(c) $$\mathcal{L}[e^{2t}u(-t) \oplus e^{-t}u(t)] = \frac{1}{s+1}\left(\frac{-1}{s-2}\right)\bigg|_{s=-s}$$

$$= \frac{1}{(s+1)(s+2)}, \qquad \sigma > -1 \cap \sigma > -2$$

$$= \frac{1}{(s+1)(s+2)}, \qquad \sigma > -1$$

The inverse Laplace transform will provide an alternative approach for correlation. For example, when finding $x(t) \oplus y(t)$ we have the choice of evaluating:

$$[x(t) \oplus y(t)] = \int_{-\infty}^{\infty} y(p)x(p-t)\,dp \qquad \text{or for example}$$

$$[e^{2t}u(-t) \oplus e^{-t}u(t)] = \mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)}\right], \qquad \sigma > -1$$

Close examination of Example 5-6(a), (b), and (c) would lead to general relations concerning symmetry of the poles for the Laplace transform of correlation functions, which will be developed later in the chapter.

5-3 THE INVERSE TWO-SIDED LAPLACE TRANSFORM

The uniqueness theorem for the inverse two-sided Laplace transform states:

$$\mathcal{L}^{-1}[\mathcal{L}(f(t))] = f(t) \tag{5-8}$$

and in this section we discuss two techniques for finding inverses. These are:

1. the use of partial fraction expansions plus table reference.
2. the classical evaluation from the formal definition of the inverse using the residue theory from complex variables.

5-3-1 Inverse Transforms Using Partial Fraction

We confine our discussion to Laplace transforms which are the ratio of two polynomials in s:

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

where the order of $D(s)$ is at least one order higher than $N(s)$. There is associated with $F(s)$ a region of convergence, $\sigma_1 < \sigma < \sigma_2$, where $D(s)$ contains two consecutive poles at $s = \sigma_1 + j\omega_1$ and $s = \sigma_2 + j\omega_2$ and we assume any cancellation of poles by zeros has been carried out. From Section 5-1 we know that a pole of $F(s)$ at $s = -s_1$ where $\text{Re}(-s_1) < \sigma_1$ contributes $Ae^{-s_1 t}u(t)$ if s_1 is real and a pole of $F(s)$ at $s = -s_2$ contributes $Be^{-s_2 t}u(-t)$ if s_2 is real and $-s_2 > \sigma_2$. We now use previously learned partial fraction theory to find inverses. A short general table of two-sided transforms is given in Table 5-2, which we will utilize when finding inverses.

TABLE 5-2 SOME FUNDAMENTAL TWO-SIDED LAPLACE TRANSFORMS

$F(s)$	Region of convergence	$f(t)$
$\frac{A}{s+\alpha}$	$\sigma > -\alpha$	$Ae^{-\alpha t}u(t)$
$\frac{A}{(s+\alpha)}$	$\sigma < -\alpha$	$-Ae^{-\alpha t}u(t)$
$\frac{A}{(s+\alpha)^2}$	$\sigma > -\alpha$	$Ate^{-\alpha t}u(t)$
$\frac{A}{(s+\alpha)^2}$	$\sigma < -\alpha$	$-Ate^{-\alpha t}u(-t)$
$\frac{A_1 s + A_2}{(s+\alpha)^2+\beta^2}$	$\sigma > -\alpha$, α real, β real and positive	$e^{-\alpha t}\left[A_1 \operatorname{Cos}\beta t + \frac{A_2 - \alpha A_1}{\beta}\operatorname{Sin}\beta t\right]u($
$\frac{A_1 s + A_2}{(s+\alpha)^2+\beta^2}$	$\sigma < -\alpha$	$-e^{-\alpha t}\left[A_1 \operatorname{Cos}\beta t + \frac{A_2 - \alpha A_1}{\beta}\operatorname{Sin}\beta t\right]u(-$

EXAMPLE 5-7

Find the inverse Laplace transforms of the following functions using partial fractions:

(a) $F_1(s) = \dfrac{3s+2}{(s+1)^2(s-2)}, \qquad -1 < \sigma < 2$

(b) $F_2(s) = \dfrac{3s+2}{(s+1)^2(s-2)}, \qquad \sigma < -1$

(c) $F_3(s) = \dfrac{3s+2}{(s+1)^2(s-2)}, \qquad \sigma > 2$

(d) $F_4(s) = \dfrac{2s+3}{(s+4)(s^2+2s+3)}, \qquad -4 < \sigma < -1$

Solution

(a)
$$F_1(s) = \frac{3s+2}{(s+1)^2(s-2)}, \qquad -1 < \sigma < 2$$
$$= \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s-2}$$

where $C = 8/(3)^2 = 0.89$ and $B = -1/-3 = 0.33$. Letting $s = 0$, we find:

$$\frac{2}{-2} = A + 0.33 + \frac{0.89}{-2}$$

Therefore
$$A = -1 - 0.33 + 0.44$$
$$= -0.89$$

$$F_1(s) = \frac{-0.89}{s+1} + \frac{0.33}{(s+1)^2} + \frac{0.89}{s-2}, \qquad -1 < \sigma < 2$$

and from Table 5-2:

$$f_1(t) = (-0.89e^{-t} + 0.33te^{-t})u(t) - 0.89e^{2t}u(-t)$$

$F_1(s)$ and $f_1(t)$ are shown in Figure 5-2(a).

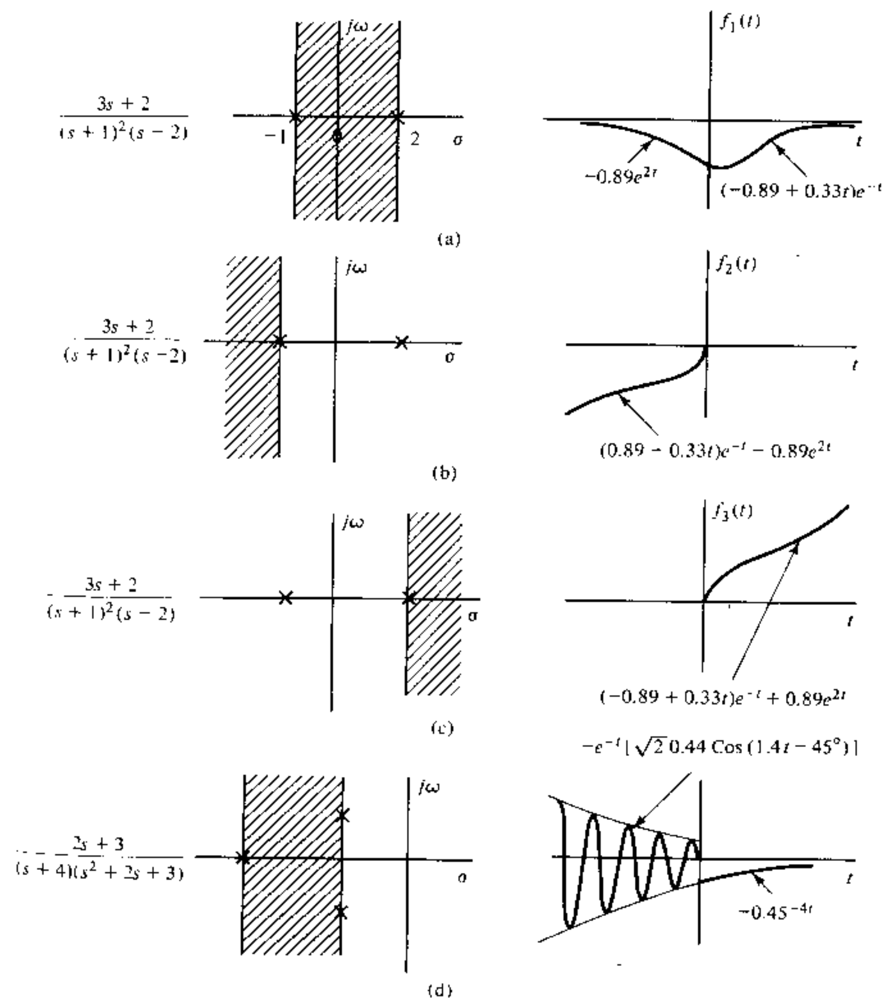


Figure 5-2 The transforms and their inverses for Example 5-6.

(b)

$$F_2(s) = \frac{3s+2}{(s+1)^2(s-2)}, \qquad \sigma < -1$$

$$= \frac{-0.89}{s+1} + \frac{0.33}{(s+1)^2} + \frac{0.89}{s-2}, \qquad \sigma < -1$$

Therefore $\quad f_2(t) = [0.89e^{-t} - 0.33te^{-t} - 0.89e^{2t}]u(-t)$

$F_2(s)$ and $f_2(t)$ are shown in Figure 5-2(b).

(c)

$$F_3(s) = \frac{3s+2}{(s+1)^2(s-2)}, \qquad \sigma > 2$$

Therefore $\quad f_3(t) = [-0.89e^{-t} + 0.33te^{-t} + 0.89e^{2t}]u(t)$

$F_3(s)$ and $f_3(t)$ are shown in Figure 5-2(c).

(d) $F_4(s) = \dfrac{2s+3}{(s+4)(s^2+2s+3)}, \qquad -4 < \sigma < -1$

$$= \frac{2s+3}{(s+4)[(s+1)^2+2]}$$

$$= \frac{2s+3}{(s+4)(s+1+j1.4)(s+1-j1.4)}$$

$$= \frac{-0.45}{s+4} + \frac{A_1}{s+1+j1.4} + \frac{A_1^*}{s+1-j1.4}, \qquad -4 < \sigma < -1$$

where $\quad A_1 = \dfrac{2(-1-j1.4)+3}{(3-j1.4)(-j2.83)} = 0.22 + j0.22 \quad$ and

$$f_4(t) = -0.45e^{-4t}u(t) - e^{-t}(0.44\,\text{Cos}\,1.4t + 0.44\,\text{Sin}\,1.4t)u(-t)$$

$F_4(s)$ and $f_4(t)$ are plotted in Figure 5-2(d).

From Example 5-7 it is seen that all the work required for evaluating inverse two-sided transforms by partial fractions was already mastered for the one-sided case. Now poles to the right of σ contribute to the function for negative time and pick up a minus sign.

5-3-2 Inverse Two-sided Laplace Transforms Using Residues

The Appendix on complex variables lists some of the more important results pertaining to system analysis and they will be utilized in this section. The inverse Laplace transform of $F(s) = N(s)/D(s)$ where the order of $D(s)$ is more than $N(s)$, with the region of convergence $\sigma_1 < \sigma < \sigma_2$ is defined as:

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\substack{\sigma-j\infty \\ C}}^{\sigma+j\infty} F(s)e^{+st}\,ds \qquad (5\text{-}9)$$

where C is the straight line $s(\theta) = \sigma + j\theta$, $-\infty < \theta < \infty$. The contour C is shown in Figure 5-3(a). We now discuss the evaluation of Equation 5-9 for the cases of t positive and negative.

For $t > 0$

Consider closing the contour C to the left with contour C_1 as shown in Figure 5-3(b). For $t > 0$, $F(s)e^{+st} \to 0$ at all points on C_1 and by Jordan's lemma $\int_{C_1} F(s)e^{+st}\,ds = 0$.

Therefore $\quad f(t) = \dfrac{1}{2\pi j}\displaystyle\int_{\substack{\sigma-j\infty \\ C}}^{\sigma+j\infty} F(s)e^{st}\,ds + \frac{1}{2\pi j}\int_{C_1} F(s)e^{st}\,ds$

$$= \frac{1}{2\pi j}\oint_{C+C_1} F(s)e^{st}\,ds$$

$$= \Sigma\,[\text{residues of the poles of } F(s)e^{st} \text{ to the left of } \sigma] \qquad (5\text{-}10)$$

For $t < 0$

Consider closing the contour C to the right with contour C_2 as shown in Figure 5-3(c). For $t < 0$, $F(s)e^{st} \to 0$ at all points on C_2 and by Jordan's lemma $\int_{C_2} F(s)e^{st}\,ds = 0$.

Therefore $\quad f(t) = \dfrac{1}{2\pi j}\displaystyle\int_{\substack{\sigma-j\infty \\ C}}^{\sigma+j\infty} F(s)e^{st}\,ds + \frac{1}{2\pi j}\int_{C_2} F(s)e^{st}\,ds$

$$= \frac{1}{2\pi j}\oint_{C+C_2} F(s)e^{st}\,ds$$

$$= -\Sigma\,[\text{residues of the poles of } F(s)e^{st} \text{ to the right of } \sigma] \qquad (5\text{-}11)$$

The minus sign occurs because the closed contour $C + C_2$ is traversed in a

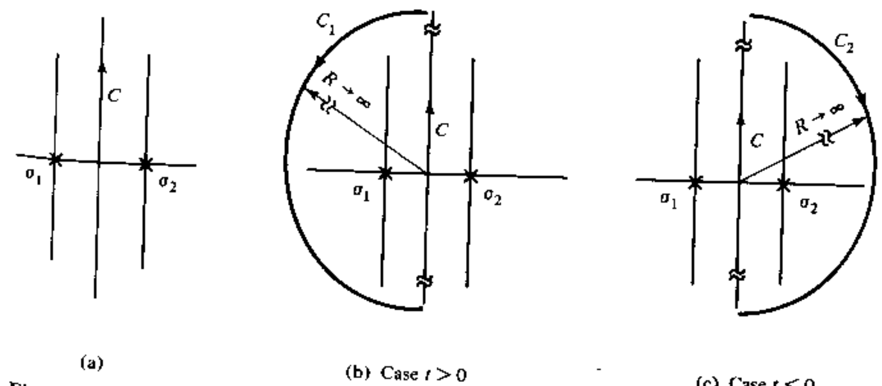


Figure 5-3 (a) The contour C for finding the inverse Laplace transform; (b) closing C with C_1 for $t > 0$; (c) closing C with C_2 for $t < 0$.

clockwise manner. Summarizing, we conclude:

> If $F(s) = N(s)/D(s)$, $\sigma_1 < \sigma < \sigma_2$ where the order of $D(s)$ is higher than $N(s)$, then
>
> *For $t > 0$*
>
> $$f(t) = \Sigma[\text{residues of the poles of } F(s)e^{st} \text{ to the left of } \sigma]$$
>
> *For $t < 0$*
>
> $$f(t) = -\Sigma[\text{residues of the poles of } F(s)e^{st} \text{ to the right of } \sigma]$$

In addition, the uniqueness theorem for the Laplace transform is:

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \left[\int_{-\infty}^{\infty} f(p) e^{-sp}\, dp \right] e^{st}\, ds \qquad (5\text{-}12)$$

which says that a time function and its Laplace transform constitute a unique pair.

We evaluate some previously considered inverses using residue theory.

EXAMPLE 5-8

Using residues, find the inverse Laplace transforms of the following functions:

(a) $F_1(s) = \dfrac{-1}{s+2}, \qquad \sigma < -2$

(b) $F_2(s) = \dfrac{1}{s+2}, \qquad \sigma > -2$

(c) $F_3(s) = \dfrac{3s+2}{(s+1)^2(s-2)}, \qquad -1 < \sigma < 2$

(d) $F_4(s) = \dfrac{3s+2}{(s+1)^2(s-2)}, \qquad \sigma < -1$

(e) $F_5(s) = \dfrac{3s+2}{(s+1)^2(s-2)}, \qquad \sigma > 2$

(f) $F_6(s) = \dfrac{2s+3}{(s+4)(s^2+2s+3)}, \qquad -4 < \sigma < -1$

Solution

(a) As shown in Figure 5-4(a) $C + C_1$ enclose no poles of $F(s)$, and therefore $f(t) = 0$, if $t > 0$.
For $t < 0$, $C + C_2$ encloses the pole at $s = -2$

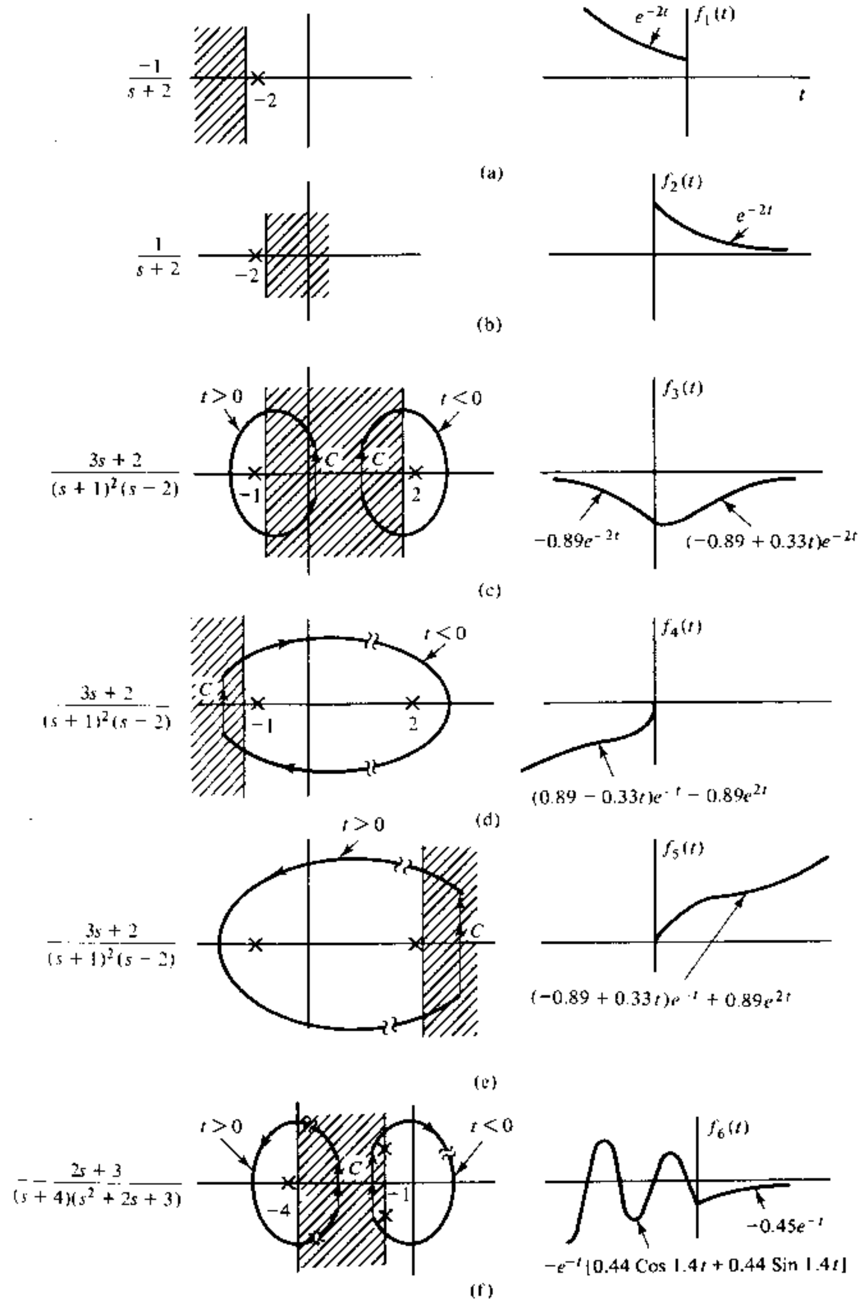


Figure 5-4 The inverse transforms for Example 5-7.

and thus

$$f_1(t) = -\left[\text{residue of pole at } s = -2 \text{ for } \frac{-1}{s+2}e^{st}\right]$$

$$= -\left[(s+2)\frac{-1}{s+2}e^{st}\Big|_{s=-2}\right]$$

$$= e^{-2t}$$

Therefore

$$f_1(t) = e^{-2t}u(-t)$$

(b) $f_2(t) = 0$, $t < 0$ as $C + C_2$ encloses no poles in Figure 5-4(b). For $t > 0$,

$$f_2(t) = \left[\text{residue of pole at } s = -2 \text{ for } \frac{1}{s+2}e^{st}\right]$$

Therefore $f_2(t) = e^{-2t}u(t)$

(c) $F_3(s) = \dfrac{3s+2}{(s+1)^2(s-2)}, \qquad -1 < \sigma < 2$

as shown in Figure 5-4(c).

For $t > 0$

$$f_3(t) = [\text{residue of the second-order pole at } s = -1 \text{ for } F(s)e^{st}]$$

$$= \frac{d}{ds}\left[\frac{3s+2}{s-2}e^{st}\right]_{s=-1}$$

$$= \frac{-1}{-3}te^{-t} + e^{st}\frac{(s-2)(3)-(3s+2)1}{(s-2)^2}\Bigg|_{s=-1}$$

$$= 0.33te^{-t} - 0.89e^{-t}$$

For $t < 0$

$$f_3(t) = -\,[\text{residue of pole at } s = 2 \text{ for } F(s)e^{st}]$$

$$= -\frac{3(2)+2}{(2+1)^2}e^{2t}$$

$$= -0.89e^{2t}$$

Therefore $f_3(t) = (-0.89 + 0.33t)e^{-t}u(t) - 0.89e^{2t}u(-t)$

This result agrees with Example 5-7(a) where the inverse was found by partial fractions.

(d) $F_4(s) = \dfrac{3s+2}{(s+1)^2(s-2)}, \qquad \sigma < -1$

as shown in Figure 5-4(d).

$$f_4(t) = 0 \text{ for } t > 0$$

since there are no poles to the left of σ.

For $t < 0$

$$f_4(t) = -\,[\text{sum of residues of poles of } F(s)e^{st}]$$

$$= (-0.33te^{-t} + 0.89e^{-t} - 0.89e^{2t})u(-t)$$

which agrees with Example 5-7(b).

(e) $F_5(s) = \dfrac{3s+2}{(s+1)^2(s-2)}, \qquad \sigma > 2$

as shown in Figure 5-4(e).

$$f_5(t) = 0, \qquad t < 0$$

since there are no poles to the right of σ.

For $t > 0$

$$f_5(t) = [\text{sum of the residues of the poles of } F(s)e^{st}]$$

Therefore $f_5(t) = (0.33te^{-t} - 0.89e^{-t} + 0.89e^{2t})u(t)$

(f) $F_6(s) = \dfrac{2s+3}{(s+4)(s^2+2s+3)}, \qquad -4 < \sigma < -1$

as shown in Figure 5-4(f).

$$= \frac{2s+3}{(s+4)(s+1+j1.4)(s+1-j1.4)}$$

For $t > 0$

$$f_6(t) = [\text{residue of the pole at } s = -4 \text{ for } F(s)e^{st}]$$

$$= -\frac{5}{11}e^{-4t}$$

$$= -0.45e^{-4t}$$

For $t < 0$

$$f_6(t) = -\,[\text{sum of the residues of the poles at } s = -1 - j1.4 \text{ and } -1 + j1.4]$$

$$= -\left[\frac{2(-1-j1.4)+3}{(3-j1.4)(-j2.8)}e^{(-1-j1.4)t} + r_1^* e^{(-1+j1.4)t}\right]$$

where

$$r_1 = 0.22 + j0.22$$

$$f_6(t) = -[(0.22 + j0.22)e^{(-1-j1.4)t} + (0.22 - j0.22)e^{(-1+j1.4)t}]$$

$$= -[e^{-t}(0.44 \text{ Cos } 1.4t + 0.44 \text{ Sin } 1.4t)]$$

Finally, as in Example 5-7(d) we obtain:

$$f_6(t) = -0.45e^{-4t}u(t) + [-0.44e^{-t}(\text{Cos } 1.4t + \text{Sin } 1.4t)]u(-t)$$

5-3-3 Complex Convolution

Complex convolution is an excellent illustration of the use of residues in transform theory. We now derive the Laplace transform for the product of two functions and illustrate complex convolution for some simple cases.

EXAMPLE 5-9

Prove:

$$\mathcal{L}[x(t)y(t)] = \frac{1}{2\pi j}\int_{\substack{a-j\infty \\ C}}^{a+j\infty} Y(p)X(s-p)\,dp, \qquad \sigma_{s1} < \text{Re}(s) < \sigma_{s2}$$

Pay particular attention to the relation between a and s.

Solution

Let $\quad x(t) \leftrightarrow X(s), \qquad \sigma_{x1} < \sigma < \sigma_{x2}$

and $\quad y(t) \leftrightarrow Y(s), \qquad \sigma_{y1} < \sigma < \sigma_{y2}$

The product function $x(t)y(t)$ must then have a region of convergence $\sigma_{x1} + \sigma_{y1} < \sigma < \sigma_{x2} + \sigma_{y2}$. This is so because $\int_0^\infty |x(t)|e^{-\sigma t}\,dt$ exists for $\sigma > \sigma_{x1}$ and $\int_0^\infty |y(t)|e^{-\sigma t}\,dt$ exists for the $\sigma > \sigma_{y1}$, and from the concept of the region of convergence then $\int_0^\infty |x(t)y(t)|e^{-\sigma t}\,dt$ exists for $\sigma > \sigma_{x1} + \sigma_{y1}$. Similarly, the behavior of $x(t)y(t)$ for $t < 0$ implies $\sigma < \sigma_{x2} + \sigma_{y2}$.

$$\mathcal{L}[x(t)y(t)] = \int_{-\infty}^{\infty} x(t)y(t)e^{-st}\,dt$$

$$= \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi j}\int_{\substack{\sigma_y-j\infty \\ C_1}}^{\sigma_y+j\infty} Y(p)\,e^{pt}\,dp\right]e^{-st}\,dt$$

where $\sigma_{y1} < \sigma_y < \sigma_{y2}$. Assuming it is permissible to interchange the order of integration, we have

$$\mathcal{L}[x(t)y(t)] = \frac{1}{2\pi j}\int_{\substack{\sigma_y-j\infty \\ C_1}}^{\sigma_y+j\infty} Y(p)\left[\int_{-\infty}^{\infty} x(t)e^{-(s-p)t}\,dt\right]dp$$

$$= \frac{1}{2\pi j}\int_{\substack{a-j\infty \\ C}}^{a+j\infty} Y(p)X(s-p)\,dp$$

$$= X(s)*Y(s)$$

We must be very careful with the allowable values of a. $X(s)*Y(s)$ exists for $\sigma_{x1} + \sigma_{y1} < \sigma < \sigma_{x2} + \sigma_{y2}$. In addition, to satisfy $Y(p)$ a must be such that $\sigma_{y1} < a < \sigma_{y2}$ and to satisfy $X(s-p)$ a must be such that $\sigma_{x1} < \text{Re}(s) - a < \sigma_{x2}$ or $-\sigma_{x2} + \text{Re}(s) < a < -\sigma_{x1} + \text{Re}(s)$. To clarify any confusion about this, we now solve a few problems.

EXAMPLE 5-10

(a) Given:

$$x(t) = e^{2t}u(t) \quad \text{and} \quad y(t) = tu(t)$$

use complex convolution to find the Laplace transform of $x(t)y(t)$.

(b) Given:

$$x(t) = e^{t}u(-t) + u(t) \quad \text{and} \quad y(t) = e^{2t}u(-t)$$

use complex convolution to find the Laplace transform of $x(t)y(t)$.

Solution

(a) $\quad x(t) = e^{2t}u(t)$

therefore $\quad X(s) = \dfrac{1}{s-2}, \qquad \sigma > 2$

$\quad y(t) = tu(t)$

thus $\quad Y(s) = \dfrac{1}{s^2}, \qquad \sigma > 0$

$\overline{x(t)y(t)}$ will exist for $\sigma > 2$.

$$\overline{x(t)y(t)} = \frac{1}{2\pi j}\int_{\substack{a-j\infty \\ c}}^{a+j\infty} \frac{1}{p^2}\,\frac{1}{s-p-2}\,dp$$

where $\text{Re}(s) > 2$ and there is a first-order pole at $p = s - 2$. We note that $\text{Re}(s-p) > 2$, and therefore $\text{Re}(s) - a > 2$ or $a < \text{Re}(s) - 2$, and in addition, $a > 0$. Figure 5-5(a) shows a plot of the two poles at $p = 0$ and $p = s - 2$, and since $\text{Re}(s) > 2$, the pole at $s - 2$ is always to the right of the pole at $p = 0$. Using the inside–outside theorem, we may close C to the left or right.

Therefore $\quad \overline{x(t)y(t)} =$ [residue of the pole at $p = 0$ (when we close the contour to the left)]

$$= \frac{d}{dp}\left[\frac{1}{s-p-2}\right]_{p=0}$$

$$= -(s-p-2)^{-2}(-1)\big|_{p=0}$$

$$= \frac{1}{(s-2)^2}$$

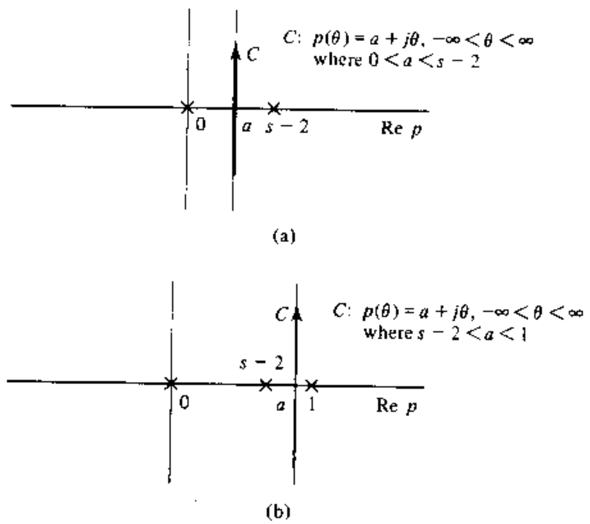


Figure 5-5 (a) Poles of $Y(p)$ and $X(s-p)$ for Example 5-9(a); (b) poles of $X(p)$ an $Y(s-p)$ for Example 5-9(b).

Also,

$$\overline{x(t)y(t)} = -[\text{residue of the pole at } p = s - 2 \text{ (when we close the contour to the right)}]$$

$$= -\left[(p-(s-2))\frac{1}{p^2[-(p-s+2)]}\right]\Bigg|_{p=s-2}$$

$$= \frac{1}{(s-2)^2}$$

Therefore $\quad \mathcal{L}[x(t)y(t)] = \dfrac{1}{(s-2)^2}, \qquad \sigma > 2$

which can easily be verified directly.

(b)

$$x(t) = e^t u(-t) + u(t)$$

and

$$X(s) = \frac{-1}{s-1} + \frac{1}{s}, \qquad 0 < \sigma < 1$$

$$= \frac{-1}{s(s-1)}, \qquad 0 < \sigma < 1$$

$$y(t) = e^{2t}u(-t)$$

and

$$Y(s) = \frac{-1}{s-2}, \qquad \sigma < 2$$

Using complex convolution, we obtain:

$$\overline{x(t)y(t)} = \frac{1}{2\pi j}\int_{\substack{a-j\infty \\ c}}^{a+j\infty} \frac{-1}{p(p-1)}\,\frac{-1}{s-p-2}\,dp$$

where the transform exists for $-\infty < \text{Re}(s) < 3$ and a for the path is such that $0 < a < 1$ and $\text{Re}(s - p) < 2$ or $\text{Re}(s) - 2 < a$ for any acceptable s. Figure 5-5(b) shows a plot of the poles of $-1/p(p-1)(p - s + 2)$ where $(s-2) < a < 1$ for any acceptable $\text{Re}(s)$. Again, we can find $x(t)y(t)$ in two ways by closing C to the left or right

$$\overline{x(t)y(t)} = [\text{residues of the poles at } p = 0 \text{ and } p = s - 2]$$

$$= \frac{-1}{-1(-s+2)} + \frac{-1}{(s-2)(s-3)}$$

$$= \frac{-s+3-1}{(s-2)(s-3)} = \frac{-1}{s-3}$$

Also,

$$\overline{x(t)y(t)} = -[\text{residue of the pole at } p = 1]$$

$$= -\left(\frac{-1}{-s+3}\right),$$

$$= \frac{-1}{s-3} \qquad \text{as before}$$

Therefore $\quad \overline{x(t)y(t)} = \dfrac{-1}{s-3}, \qquad -\infty < \text{Re}(s) < 3$

The integration is very tricky. We carefully note the pole at $p = s - 2$ is always to the left of the pole at $p = 1$.

Although using such tricky mathematics to find the Laplace transform of easy product functions may seem cumbersome, trying to understand the principle is worthwhile. Historically, complex convolution was instrumental in the development of the fast Fourier transform.

A very important application of complex convolution is to relate the Laplace transform of a continuous function $f(t)$ to the Laplace transform of $f^*(t) = \tau\,\Sigma_{-\infty}^{\infty} f(n\tau)\delta(t - n\tau)$. We will carry this out for causal functions and leave the general case as an exercise.

EXAMPLE 5-11

Given $f(t)$ has a Laplace transform $F(s)$ with $\sigma > \sigma_1$, use complex convolution to find the Laplace transform of:

$$f^*(t) = \tau\sum_{0}^{\infty} f(n\tau)\delta(t - n\tau)$$

Solution. Using the sifting property of delta functions $f(t)\delta(t - a) = f(a)\delta(t - a)$, we obtain:

$$f^*(t) = \sum_{0}^{\infty} \tau f(n\tau)\delta(t - n\tau)$$

$$= \tau f(t) \sum_0^\infty \delta(t - n\tau)$$

$$\mathcal{L}\left[\sum_0^\infty \delta(t - n\tau)\right] = 1 + e^{-\tau s} + e^{-2\tau s} + \cdots + e^{-n\tau s} + \cdots$$

$$= \frac{1}{1 - e^{-\tau s}}, \qquad \sigma > 0$$

Using complex convolution, we have:

$$\mathcal{L}\left[\tau f(t) \sum_0^\infty \delta(t - n\tau)\right] = \frac{1}{2\pi j} \int_C \frac{\tau F(p)}{1 - e^{-\tau(s-p)}}\, dp$$

If $C = \sigma + j\beta$, $-\infty < \beta < \infty$, then σ must be such that $\sigma > \sigma_1$, $\mathrm{Re}(s) - \sigma >$
for $\mathrm{Re}(s) > 0$.

EXAMPLE 5-12
Given

$$f(t) = 3e^{+4t}u(t)$$

and $f^*(t)$ is found by sampling $f(t)$ every 0.01 s and approximating it by

$$f^*(t) = \sum_{n=0}^\infty 0.01 f\left(\frac{n}{100}\right) \delta\left(t - \frac{n}{100}\right)$$

find $F^*(s)$.

Solution

$$F(s) = \frac{3}{s+4}, \qquad \sigma > -4$$

$$\mathcal{L}\left[\sum_0^\infty \delta\left(t - \frac{n}{100}\right)\right] = \frac{1}{1 - e^{-0.01s}}, \qquad \sigma > 0$$

and

$$F^*(s) = F(s) * \frac{0.01}{1 - e^{-0.01s}}$$

$$= \frac{1}{2\pi j} \int_C \frac{0.03}{p + 4} \frac{1}{1 - e^{-0.01(s-p)}}\, dp$$

where $C = \sigma + j\beta$ is such that $\sigma > -4$ and $\mathrm{Re}(s) - \sigma > 0$ or $\sigma < \mathrm{Re}(s)$ f
any $\mathrm{Re}(s) > 0$. The path C is always to the right of $\mathrm{Re}(p) = -4$ and to t
left of the poles due to $1 - e^{-0.01(s-p)}$. We should make a pole zero sket
similar to that of Figure 5-5(b).

Finding $F^*(s)$ by closing C to the left, we obtain:

$$F^*(s) = [\text{residue of the pole at } p = -4]$$

$$= \frac{0.03}{1 - e^{-0.01s}e^{-0.04}}$$

The form of $F^*(s)$ is somewhat unwieldy and from introductory complex variables we can see that $F^*(s)$ contains an infinite number of poles, all with the same real part. (Can you find them?) However, an important continuation of this problem is considered in Chapter 6 when the Z transform is found for the discrete function obtained by sampling $f(t)$ every τ seconds.

5-4 LINEAR SYSTEMS WITH RANDOM AND SIGNAL PLUS NOISE INPUTS

In Chapter 3 we found the output autocorrelation function and cross-correlation function of the input with the output when the input to a LTIC continuous system is a noise waveform with autocorrelation function $R_{xx}(\tau)$:

$$R_{yy}(\tau) = C_{hh}(\tau) * R_{xx}(\tau) \tag{5-13}$$

and

$$R_{xy}(\tau) = R_{xx}(\tau) \oplus h(\tau) \tag{5-14a}$$

or

$$R_{xy}(\tau) = h(\tau) * R_{xx}(\tau) \tag{5-14b}$$

These results are shown schematically in Figure 5-6(a). Let us denote $\mathcal{L}[R_{xx}(\tau)]$ by $S_{xx}(s)$, $\mathcal{L}[R_{xy}(\tau)]$ by $S_{xy}(s)$, $\mathcal{L}[R_{yx}(\tau)]$ by $S_{yx}(s)$, $\mathcal{L}[R_{yy}(\tau)]$ by $S_{yy}(s)$, and $\mathcal{L}[C_{hh}(\tau)]$ by $T(s)$. We call $S_{xx}(s)$ the power spectral density of $x(t)$, $S_{xy}(s)$ the cross-spectral density of $x(t)$ and $y(t)$, $S_{yx}(s)$ the cross-spectral density of $y(t)$ and $x(t)$, and $S_{yy}(s)$ the power spectral density of $y(t)$ and $T(s)$ the power transfer function. Technically, these names are more meaningful to physical interpretation when the Fourier transforms of $R_{xx}(\tau)$, $R_{xy}(\tau)$, $R_{yx}(\tau)$, and $C_{hh}(\tau)$ are used.

We now find expressions for these spectral quantities using the convolution and correlation theorems:

$$S_{yy}(s) = \mathcal{L}[h(t) \oplus h(t)]S_{xx}(s)$$

$$= [H(s)H(-s)]S_{xx}(s) \tag{5-15}$$

where

$$T(s) = H(s)H(-s)$$

$$S_{xy}(s) = \mathcal{L}[R_{xx}(\tau) \oplus h(\tau)]$$

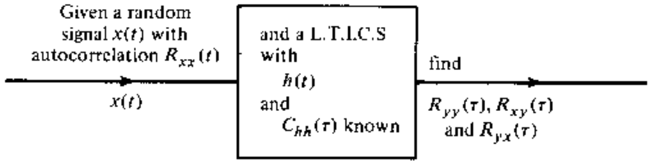
$$= H(s)S_{xx}(-s) \tag{5-16}$$

or

$$S_{xy}(s) = \mathcal{L}[h(\tau) * R_{xx}(\tau)]$$

$$= H(s)S_{xx}(s) \tag{5-17}$$

These results are tabulated in Figure 5-6b. It can be shown that since $R_{xx}(\tau)$ is even, $S_{xx}(s)$ and $S_{xx}(-s)$ are equivalent. Before applying these formulas, we develop some symmetry properties for spectral functions.



(a) Time-Domain Results

$$R_{yy}(\tau) = C_{hh}(\tau) * R_{xx}(\tau)$$

$$R_{xy}(\tau) = R_{xx}(\tau) \oplus h(\tau)$$

$$= h(\tau) * R_{xx}(\tau)$$

$$R_{yx}(\tau) = h(\tau) \oplus R_{xx}(\tau)$$

(b) Transform Results

$L[h(t)] = H(s)$, $\quad L[C_{hh}(\tau)] = H(s)H(-s) = T(s)$, $\quad L[R_{xx}(\tau)] = S_{xx}($

$S_{yy}(s) = S_{xx}(s)T(s)$

$S_{xy}(s) = H(s)S_{xx}(s)$ or $H(s)S_{xx}(-s)$

$S_{yx}(s) = H(-s)S_{xx}(s) = S_{xy}(-s)$

Figure 5-6 (a) The time-domain results for a system with a random input; (b) transform results.

5-4-1 Properties of Spectral Functions

For $S_{xx}(s)$

It is easy to show $S_{xx}(s) = S_{xx}(-s)$. By definition:

$$S_{xx}(s) = \int_{-\infty}^{\infty} R_{xx}(\tau)e^{-s\tau}\, d\tau$$

Let $\tau = -p$

therefore
$$S_{xx}(s) = \int_{\infty}^{-\infty} R_{xx}(-p)e^{-s(-p)}\, -dp$$

$$= \int_{-\infty}^{\infty} R_{xx}(p)e^{sp}\, dp,$$

since
$$R_{xx}(p) = R_{xx}(-p)$$

and
$$S_{xx}(s) = S_{xx}(-s) \quad (5\text{-}18)$$

This implies that if $S_{xx}(s)$ contains a pole at $s = s_p$, it must also contain a pole at $s = -s_p$, and similarly, if $S_{xx}(s)$ contains a zero at $s = s_z$, it must also contain a zero at $s = -s_z$.

This is also clear from the time domain. If an even function contains $e^{\alpha t}u(t)$, then it must also contain $e^{-\alpha t}u(-t)$. The same is also true for $C_{hh}(\tau)$ where $C_{hh}(\tau) = h(\tau) \oplus h(\tau)$.

$$\mathcal{L}[h(\tau) \oplus h(\tau)] = H(s)H(-s)$$

$$= T(s)$$

$$T(s) = T(-s) \quad (5\text{-}19)$$

Since $S_{xx}(s)$ and $T(s)$ contain product terms such as $(s - s_p)(s + s_p) = (s^2 - s_p^2)$ in the denominator and $(s - s_z)(s + s_z) = (s^2 - s_z^2)$ in the numerator, then both the numerator and denominator will be real even polynomials in s.

$$S_{xx}(s) \text{ or } T(s) = \frac{b_m s^{2m} + b_{m-2}s^{2m-2} + \cdots + b_0}{a_n s^{2n} + a_{n-2}s^{2n-2} + \cdots + a_0} \quad (5\text{-}20)$$

For $S_{yx}(s)$

Since $R_{xy}(\tau) = R_{yx}(-\tau)$, it is easy to show that $S_{xy}(s) = S_{yx}(-s)$. By definition:

$$S_{xy}(s) = \int_{-\infty}^{\infty} R_{xy}(\tau)e^{-s\tau}\, d\tau$$

Let $p = -\tau$

therefore
$$S_{xy}(s) = \int_{\infty}^{-\infty} R_{xy}(-p)e^{sp}\, -dp$$

$$= \int_{-\infty}^{\infty} R_{xy}(-p)e^{sp}\, dp$$

$$= \int_{-\infty}^{\infty} R_{yx}(p)e^{sp}\, dp$$

$$= S_{yx}(-s)$$

and
$$S_{yx}(s) = H(-s)S_{xx}(s) \quad (5\text{-}21)$$

This implies that if $S_{xy}(s)$ has a pole at s_p or a zero at s_z, then $S_{yx}(s)$ has a pole at $-s_p$ or a zero at $-s_z$ and vice versa.

Summarizing the main properties of spectral functions, we have:

Property 1

$$S_{xx}(s) = S_{xx}(-s)$$

or
$$T(s) = T(-s)$$

where
$$T(s) = \mathcal{L}[C_{hh}(\tau)]$$

This implies a spectral function is the ratio of two even polynomials of s.

Property 2

$$S_{xy}(s) = S_{yx}(-s)\,.$$

EXAMPLE 5-13

Given the impulse response of a system is:

$$h(t) = (3e^{-t} + 2te^{-2t})u(t)$$

Use the Laplace transform to find the power transfer function $T(s)$, and hence $C_{hh}(\tau)$.

Solution

$$h(t) = (3e^{-t} + 2te^{-2t})u(t)$$

$$H(s) = \frac{3}{s+1} + \frac{2}{(s+2)^2} = \frac{3s^2 + 14s + 14}{(s+1)(s+2)^2} = \frac{3(s+3.1)(s+1.5)}{(s+1)(s+2)^2}$$

$$L[h(t) \oplus h(t)] = T(s)$$

$$= \frac{3(s+3.1)(s+1.5)}{(s+1)(s+2)^2} \times \frac{3(-s+3.1)(-s+1.5)}{(-s+1)(-s+2)^2}$$

$$= \frac{9(+1)(s+3.1)(s-3.1)(s+1.5)(s-1.5)}{(-1)(s+1)(s-1)(-1)^2[(s+2)(s-2)]^2}$$

$$= \frac{-9(s+3.1)(s-3.1)(s+1.5)(s-1.5)}{(s+1)(s-1)[(s+2)(s-2)]^2},$$

$$-1 < \sigma < 1$$

As the ratio of two even polynomials this becomes:

$$T(s) = \frac{-9(s^2 - 9.6)(s^2 - 2.25)}{(s^2-1)(s^2-4)^2} = \frac{-9[s^4 - 11.85s^2 + 21.6]}{s^6 - 9s^4 + 24s^2 - 16}, \quad -1 < \sigma < 1$$

and
$$C_{hh}(\tau) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} \frac{-9(s^2 - 9.6)(s^2 - 2.25)}{(s-1)(s+1)(s-2)^2(s+2)^2} e^{s\tau}\, ds$$

For $\tau > 0$

$$C_{hh}(\tau) = \text{(residue of pole at } -1) + \text{(residue of pole at } -2)$$

$$= \frac{-9(-8.6)(-1.25)}{(-2)(-3)^2(1)^2} e^{-\tau}\Big|_{s=-1} + \frac{d}{ds}\left[\frac{-9(s^4 - 11.85s^2 + 21.6)}{(s-1)(s+1)(s-2)^2} e^{s\tau}\right]\Bigg|_{s=-2}$$

$$= 5.4e^{-\tau} + \frac{-9(16 - 47.4 + 21.6)}{48}\tau e^{-2\tau} + e^{-2\tau}\frac{d}{ds}\left[\frac{-9(s^4 - 11.85s^2 + 21.6)}{s^4 - 4s^3 + 3s^2 + 4s - 4}\right]\Bigg|_{s=-2}$$

$$= 5.4e^{-\tau} + 1.8\tau e^{-2\tau} + 4.33e^{-2\tau}$$

Using the evenness of $C_{hh}(\tau)$, we get:

$$C_{hh}(\tau) = 5.4e^{-|\tau|} + 1.8\,|\tau|\, e^{-2|\tau|} + 4.33e^{-2|\tau|}$$

5-4-2 Deterministic Signal Plus Uncorrelated Zero-Mean Noise

Figure 5-7(a) shows a linear system with system function $H(s)$ and power transfer function $T(s) = H(s)H(-s)$. The input is $x(t) = f(t) + n(t)$, where $f(t)$ is a deterministic signal and $n(t)$ is zero-mean uncorrelated noise (i.e., $R_{fn}(\tau) = 0$) with autocorrelation function $R_{nn}(\tau)$. In Chapter 3 we found the deterministic output as:

$$g(t) = f(t) * h(t)$$

and the output noise autocorrelation function as:

$$R_{mm}(\tau) = R_{nn}(\tau) * C_{hh}(\tau)$$

and the cross-correlation of $n(t)$ with $m(t)$ as:

$$R_{nm}(\tau) = h(\tau) * R_{xx}(\tau)$$

Using the bilateral Laplace transform, we obtain:

$$G(s) = F(s)H(s) \tag{5-22}$$

$$S_{mm}(s) = S_{nn}(s)[H(s)H(-s)] \tag{5-23}$$

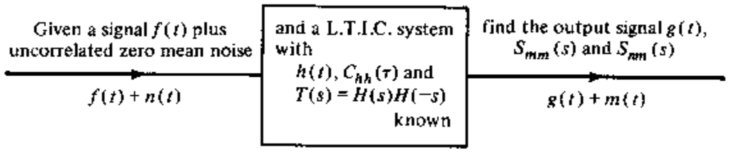
$$S_{nm}(s) = H(s)S_{nn}(s) \tag{5-24}$$

$$S_{mn}(s) = H(-s)S_{nn}(s) \tag{5-25}$$

These relations are summarized in Figure 5-7(b). We conclude this section by resolving Example 3-17 from Chapter 3 by using the bilateral Laplace transform.

EXAMPLE 5-14

Consider a linear system with impulse response $h(t) = 2e^{-3t}u(t)$ and with a deterministic input $f(t) = 3 \text{ Cos } 2t$ plus uncorrelated white noise with a mean square value of 100 whose autocorrelation function may be approximated by $R_{nn}(\tau) = 4\delta(\tau)$. Find the output signal, the output power spectral density $S_{mm}(s)$, and hence the output autocorrelation function and the input and output signal to noise ratios.



(a) Time-Domain Results (Chapter 3)

$$g(t) = f(t) * h(t)$$
$$R_{mm}(\tau) = R_{nn}(\tau) * C_{hh}(\tau)$$
$$R_{nm}(\tau) = h(\tau) * R_{nn}(\tau)$$

(b) Transform Results

$$G(s) = H(s)F(s)$$
$$S_{mm}(s) = S_{nn}(s)T(s)$$
$$S_{nm}(s) = H(s)S_{nn}(s)$$
$$S_{mn}(s) = H(-s)S_{nn}(s)$$

Figure 5-7 (a) The time-domain results for a system with a signal plus uncorrelated noise input; (b) the transform results.

Solution

The Output Signal

As in Example 3-17, using phasors, we get:

$$g(t) = \text{Re}\left[\frac{2}{j2+3} 3\angle 0^\circ e^{j2t}\right]$$
$$= 1.66 \text{ Cos } (2t - 33^\circ)$$

The Output Power Spectral Density

$$S_{nn}(s) = \overline{[4\delta(\tau)]} = 4$$
$$H(s) = \mathcal{L}\,[2e^{-3t}u(t)]$$
$$= \frac{2}{s+3}$$

and the power transfer function is:

$$T(s) = H(s)H(-s)$$
$$= \frac{2}{s+3}\,\frac{2}{-s+3}$$
$$= \frac{-4}{s^2-9}; \qquad -3 < \sigma < 3$$

$$C_{hh}(\tau) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \frac{-4}{s^2-9} e^{s\tau}\, ds$$

For $\tau > 0$

$$C_{hh}(\tau) = [\text{residue of the pole at } s = 3]$$
$$= \frac{-4}{-6} e^{-3\tau} u(\tau)$$

therefore

$$C_{hh}(\tau) = 0.67e^{-3\tau}u(\tau) + 0.67e^{3\tau}u(-\tau)$$
$$S_{mm}(s) = T(s)S_{nn}(s)$$
$$= \frac{-16}{s^2-9}, \qquad -3 < \sigma < 3$$

and

$$R_{mm}(\tau) = 2.67e^{-3|\tau|}$$

As before, we can find:

$$\left.\frac{S}{N}\right|_{\text{input}} = \frac{\left(\frac{3}{\sqrt{2}}\right)^2}{100} = 0.045$$

and

$$\left.\frac{S}{N}\right|_{\text{output}} = \frac{\left(\frac{1.66}{\sqrt{2}}\right)^2}{R_{mm}(0)} = 0.53$$

This problem was just as easy in the time domain since the assumption of white noise made the calculation:

$$R_{mm}(\tau) = C_{hh}(\tau) * R_{nn}(\tau)$$

trivial. If $R_{nn}(\tau)$ is not so simple, the transform approach is easier, and in more difficult cases finding $R_{mm}(\tau)$ as the inverse of $S_{mm}(s)$ is available as a computer program.

SUMMARY

The two-sided Laplace transform was defined as $F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}\, dt$, and if it exists it does so for a region of convergence $\sigma_1 < \text{Re}(s) < \sigma_2$. The behavior of $f(t)$ for positive time places the lower bound σ_1 on Re(s) and the behavior of $f(t)$ for negative time places the upper bound of σ_2 on Re(s). The experience of evaluating one-sided transforms may be utilized to find two-sided transforms. If $f(t) = f_1(t)u(t) + f_2(t)u(-t)$ and $F(s)$ exists, then $F(s) = F_1(s) + F_2(s)$ where $F_2(s)$ is the one-sided transform of $f_2(-t)u(t)$ with $s = -s$.

The most commonly occurring two-sided functions in linear system theory are correlation functions, whether $C_{hh}(\tau)$, the correlation of the impulse response of a system with itself, or the autocorrelation and cross-correlation functions of ergodic noise waveforms. When studying the main properties of two-sided transforms the reader should pay particular attention to the transform of correlation integrals:

$$\mathcal{L}\,[x(t) \oplus y(t)] = Y(s)X(-s)$$

and

$$\mathcal{L}\,[x(t) \oplus x(t)] = X(s)X(-s)$$

The time-domain results for a linear system with an ergodic random input whose autocorrelation function is $R_{xx}(\tau)$ are:

$$R_{yy}(\tau) = R_{xx}(\tau) * C_{hh}(\tau)$$

and

$$R_{xy}(\tau) = h(\tau) * R_{xx}(\tau)$$

In the transform domain these results are:

$$S_{yy}(s) = S_{xx}(s)\,T(s)$$

where

$$T(s) = H(s)H(-s)$$

and

$$S_{xy}(s) = H(s)S_{xx}(s)$$

where $S_{xx}(s)$ and $S_{yy}(s)$, the transforms of the autocorrelation functions, are called the power spectral densities. Using inverse transforms, we find that the correlation functions are:

$$R_{yy}(\tau) = \mathcal{L}^{-1}[S_{xx}(s)\,T(s)]$$

and

$$R_{xy}(\tau) = \mathcal{L}^{-1}[S_{xx}(s)H(s)]$$

Inverse transforms were evaluated in two ways. Any transform $F(s) = N(s)/D(s)$ where the order of $D(s)$ is at least one higher than $N(s)$ may be expanded in partial fractions and $f(t)$ is then found by table reference. For example, if $F(s) = 2/(s+2) + 4/(s-1)^2$; $-2 < \sigma < 1$, then $f(t) = 2e^{-2t}u(t) - 4te^{t}u(-t)$. Alternatively, the inverse may be found by the residue theory. From the definition of the inverse we have:

$$f(t) = \frac{1}{2\pi j}\int_{C\,\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}\,ds$$

By judiciously closing C to the left or right, we obtain:

When $t > 0$

$f(t) = \Sigma$[sum of the residues of the poles of $F(s)e^{st}$ to the left of σ]

When $t < 0$

$f(t) = -\Sigma$[sum of the residues of the poles of $F(s)e^{st}$ to the right of σ],

For example, the inverse transform of:

$$F(s) = \frac{2s^2 + 10}{(s+2)(s-1)^2}, \qquad -2 < \sigma < 1$$

is

$$f(t) = \frac{2(4) + 10}{9} e^{-2t}u(t) - \frac{d}{ds}\left[\frac{2s^2 + 10}{s+2} e^{st}\right]_{s=1} u(-t)$$

$$= 2e^{-2t}u(t) - 4te^{t}u(-t)$$

PROBLEMS

5-1. Find the two-sided Laplace transform and state the region of convergence for the following functions:

(a) $f_1(t) = (2 + 3t)u(t) - 3e^{t}u(-t)$ **(b)** $f_2(t) = (2 + 3t - 3e^{t})u(t)$
(c) $f_3(t) = (2 + 3t - 3e^{t})u(-t)$ **(d)** $f_4(t) = t^{10}e^{-2t}u(-t)$
(e) $f_5(t) = 6.2\,\mathrm{Cos}(4t - 40°)u(-t)$ **(f)** $f_6(t) = 2e^{-2|t|}$
(g) $(1 - |t|)[u(t+1) - u(t+1)]$

5-2. **(a)** Given:

$$f(t) = te^{-2t}u(t) - 4e^{t}u(-t)$$

and

$$g(t) = f(t - 4)$$

Use the shifting theorem to find $G(s)$. How does the pole zero configuration and region of convergence of $G(s)$ compare to those of $F(s)$?

(b) Plot the following functions and find their two-sided Laplace transforms:
(i) $e^{-(t-3)}u(t-3)$ **(ii)** $2(t+4)e^{-(t+4)}u(-t-4)$
(iii) $3e^{3t}u(-t+3)$

5-3. As quickly as possible find the inverse transforms of:

(a) $\dfrac{2}{s+2}$, $\sigma < -2$ **(b)** $\dfrac{2}{(s+3)^2}$, $\sigma < -3$

(c) $\dfrac{2s+3}{s^2+4s+3}$, $\sigma < -3$

5-4. Evaluate the following inverse Laplace transforms using partial fractions and table reference. Plot the time function in each case:

(a) $F_1(s) = \dfrac{s^2}{(2s+1)(s-3)^2}$, $-0.5 < \sigma < 3$

(b) $F_2(s) = \dfrac{s^2}{(2s+1)(s-3)}$, $-0.5 < \sigma < 3$

(c) $F_3(s) = \dfrac{3s+2}{s^2-9}$, $\sigma < -3$

(d) $F_4(s) = \dfrac{2s^2-1}{s^3+3s+2}$, where $f_4(t)$ is causal.

5-5. Repeat Problem 5-3 using the residue theory.

5-6. If possible, evaluate the following convolution and correlation integrals:

(a) $\delta(t-a) * \delta(t-b)$ **(b)** $\delta(t-a) \oplus \delta(t-b)$
(c) $\delta(t-b) \oplus \delta(t-a)$ **(d)** $2e^{-2t}u(t) * 2e^{-t}u(t)$
(e) $2e^{-2t}u(t) \oplus 2e^{-t}u(t)$ **(f)** $2e^{-2t}u(t) * 2e^{-t}u(-t)$
(g) $2e^{-2t}u(t) \oplus 2e^{-t}u(-t)$ **(h)** $2e^{-|t|} * 3e^{-2|t|}$
(i) $2e^{-|t|} \oplus 3e^{-2|t|}$ **(j)** $3e^{-2|t|} \oplus 2e^{-|t|}$

5-7. (a) If $x(t) \leftrightarrow X(s)$, $\quad \sigma_{x1} < \sigma < \sigma_{x2}$
and $y(t) \leftrightarrow Y(s)$, $\quad \sigma_{y1} < \sigma < \sigma_{y2}$
will the Laplace transform of $x(t)y(t)$ always exist?
(b) If $x(t) \leftrightarrow X(s)$, $\quad -5 < \sigma < 3$
and $y(t) \leftrightarrow Y(s)$, $\quad -1 < \sigma < 5$
and both denominator polynomials are of second order:
(1) for what range of Re(s) does $\overline{x(t)y(t)}$ exist
(2) sketch the poles of $X(p)Y(s-p)$ and the contour of integration for

$$\overline{x(t)y(t)} = \frac{1}{2\pi j}\int_{\substack{\sigma-j\infty \\ C}}^{\sigma+j\infty} X(p)Y(s-p)\,dp$$

5-8. (a) Use the complex convolution to find the Laplace transform of $x(t)y(t)$

where $\quad x(t) = 2te^{-t}u(-t) + e^{-4t}u(t)$

and $\quad y(t) = e^{t}u(-t) + tu(t)$

(b) Check your answer by finding the transform directly.

5-9. Given:

$$\overline{x(t)} = X(s), \qquad \sigma_{x1} < \sigma < \sigma_{x2}$$

and

$$\overline{y(t)} = Y(s), \qquad \sigma_{y1} < \sigma < \sigma_{y2}$$

(a) What are the conditions for $x(t)$ and $y(t)$ to be stable?
(b) List when the following are stable for $x(t)$ and $y(t)$ stable, and if they can unstable give a specific example for $x(t)$ and $y(t)$:
(1) $x(t)y(t)$ (2) $x(t)*y(t)$ (3) $x(t) \oplus x(t)$
(4) $x(t) \oplus y(t)$ (5) $y(t) \oplus x(t)$
(c) Sketch pole diagrams.

5-10. (a) Find the output of a system with the system function:

$$H(s) = 3/(s+2), \qquad \sigma > -2$$

when the input is:

$$x(t) = 2u(-t) + te^{-t}u(t)$$

(b) Plot $y(t)$.

5-11. (a) Prove that the mean square value of a random process with power spectr density $S_{xx}(s)$ is:

$$\overline{x^2(t)} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} S_{xx}(s)\,ds, \qquad \sigma_1 < \sigma < \sigma_2$$

$$= \Sigma[\text{residues of the poles of } S_{xx}(s) \text{ to the left of } \sigma]$$

or

$$= -\Sigma[\text{residues of the poles of } S_{xx}(s) \text{ to the right of } \sigma]$$

(b) For the continuous process:

$$S_{xx}(s) = \frac{-s^2+9}{s^4-5s^2+4}$$

find the mean square value of $x(t)$ using residue theory.

5-12. Given the input to a system with system function $H(s) = 2/(s+5)$ is essentiall

white noise with $S_{xx}(s) = 4$ and $\overline{x^2(t)} = 50$. Find:
(a) the output power spectral density and cross-power spectral density between the input and output.
(b) the output noise fluctuations $y^2(t)$ using:

$$\overline{y^2(t)} = R_{yy}(0) = \frac{1}{2\pi j}\int_{\substack{\sigma-j\infty \\ C}}^{\sigma+j\infty} S_{yy}(s)\,ds$$

$$= \Sigma[\text{residues of the poles to the left of } \sigma]$$

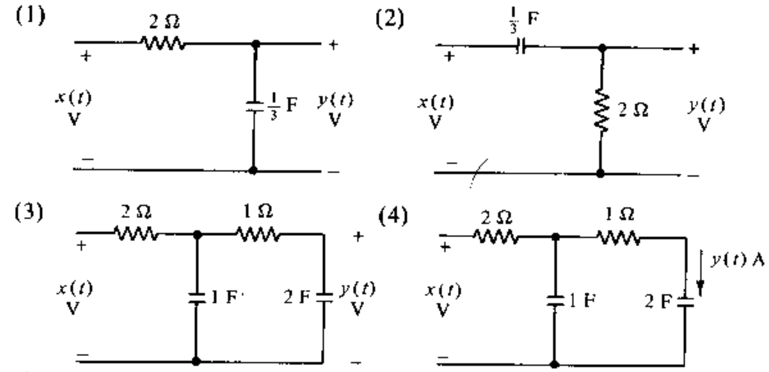
$$= -\Sigma[\text{residues of the poles to the right of } \sigma].$$

5-13. If the input to the system of Problem 5-12 consists of the deterministic signal $f(t) = 20\,\mathrm{Cos}(50t - 30°)$ plus the same white noise with $S_{nn}(s) = 4$ and $\overline{n^2(t)} = 50$, find the input and output signal to noise ratios.

5-14 The power transfer function is defined as:

$$T(s) = H(s)H(-s)$$

(a) Find the power transfer function for the following systems and plot the pole zero diagram:



(b) If the input to each of the four systems of part (a) is assumed white noise with $S_{xx}(s) = 2$ and $\overline{x^2(t)} = 100$ use residue theory to find the mean squared fluctuations at the output $\overline{y^2(t)} = R_{yy}(0)$.

5-15. (a) If the input to system (1) of part (a) of the Problem 5-14 is random noise with $R_{xx}(\tau) = 50e^{-4|\tau|}$, find the power spectral density of the output noise $S_{yy}(s)$ and the cross-spectral density of the input and output noise $S_{xy}(s)$.
(b) give a pole zero plot for $S_{xy}(s)$, $S_{yx}(s)$, and $S_{yy}(s)$.
(c) Find the signal to noise ratio at the input and output if an input signal $f(t) = 2$ is added to the input noise.

5-16. A power spectral density $S_{xx}(s)$ or power transfer function may be written as:

$$S_{xx}(s) = G(s)G(-s) \quad \text{or} \quad T(s) = H(s)H(-s)$$

where $G(s)$ or $H(s)$ have their poles and zeros in the left half plane. Design a "shaping filter" $H(s)$ that transforms white noise with $S_{xx}(s) = 2$ to noise with a power spectral density $S_{yy}(s) = (-s^2+1)/(s^4+81)$.

5-17. In the statistical communication theory, a system is often designed assuming input white noise and a prefilter is then used to transform the actual power spectral density $S_{xx}(s)$ to white noise. This is the reverse of Problem 5-16. Find $H(s)$ to convert $S_{xx}(s) = (-s^2 + 1)/(s^4 + 81)$ to white noise $S_{yy}(s) = 1$.

5-18. Which of the following functions qualify as power spectral densities?

(a) $\frac{2}{s^2-8}$ (b) $\frac{-2}{s^2-8}$

(c) $\frac{s}{s^2-8}$ (d) $\frac{-s^2+1}{s^2-8}$

(e) $\frac{-4}{(s^2-9)^2}$ (f) $\frac{1-s^2}{(s^2-9)^2}$

(g) $\frac{s^2-1}{(s^2-9)^2}$ (h) $\frac{1-s^2}{s^4-8s^2+16}$

Chapter 6

The One-sided *Z* Transform

INTRODUCTION

In Chapter 2 the time-domain analysis of LTIC discrete systems was treated. Linear difference equations of the type:

$$a_n y(n) + \cdots + a_{n-p} y(n-p) = f(n)$$

were solved classically and iteratively. It was seen that the homogeneous solution contained terms of the form $A_1\alpha^n$ or $(A_1 + A_2 n)\alpha^n$, and so on. Similarly, when $f(n)$ was of the form $(A_0 + A_1 n + A_2 n^2)\alpha^n$ the forced response was readily found by logically assuming a solution and using substitution. A general LTIC discrete system is characterized by a difference equation:

$$a_n y(n) + a_{n-1} y(n-1) + \cdots + a_{n-p} y(n-p) = b_n x(n) + \cdots + b_{n-l} x(n-l)$$

where $x(n)$ and $y(n)$ denote the input and output, respectively. The pulse response $h(n)$ was defined as the output when the input $x(n) = \delta(n)$ and pulse responses were solved for classically, by assuming a homogeneous type solution. If the input to a LTIC discrete system is $x(n) = \Sigma_k\, x(k)\delta(n-k)$ then the zero-state output $y(n)$ was found as the convolution of the input and pulse response:

$$y(n) = x(n) * h(n) = \sum_p x(p)h(n-p) \quad \text{or} \quad \sum_l h(l)x(n-l)$$

In Chapter 6 we consider the *Z* transform analysis of LTIC systems with