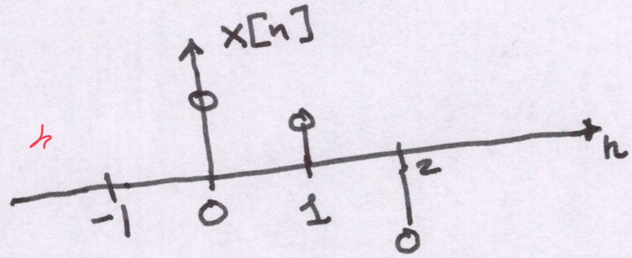
Discrete-time Fourier Transform



where $\boxed{t = nT_s}$

$$\therefore \quad e^{j\omega t} = e^{j\omega T_s n}$$

$$\boxed{\Omega = \omega T_s}$$

DTFT

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \quad -\pi < \Omega < \pi$$

$$x[n] = \frac{1}{2\pi} \int_0^{2\pi} X(\Omega) e^{j\Omega n} \, d\Omega$$

By virtue of sampling

$$X(\Omega) = X(\Omega + 2\pi k)$$

for $k = \{-2, -1, 0, 1, 2 \ldots\}$

special function

$$\delta[n] = \begin{cases} 1 & n=0 \\ 0 & \text{otherwise} \end{cases}$$

find $X(\Omega)$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\Omega n} = 1$$

inverse transform

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1) e^{j\Omega n} \, d\Omega$$

$$= \frac{1}{2\pi} \left. \frac{e^{j\Omega n}}{jn} \right|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \frac{e^{j\pi n} - e^{-j\pi n}}{jn}$$

$$= \frac{1}{2\pi} \frac{2j \sin(\pi n)}{jn}$$

$$\boxed{x[n] = \frac{\sin(\pi n)}{\pi n}} = \begin{cases} 1 & n=0 \\ 0 & \text{otherwise} \end{cases}$$

Some Prop

$$x[n] \longleftrightarrow X(\Omega)$$

1° $x[n-m_0] \longleftrightarrow X(\Omega) e^{-jm_0\Omega}$

proof

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} x[n-m_0] e^{-jn\Omega}$$

let $k = n - m_0 \Rightarrow n = k + m_0$

$$Y(\Omega) = \sum_{k=-\infty}^{\infty} x[k] e^{-jk\Omega} e^{-jm_0\Omega}$$

$$\boxed{Y(\Omega) = X(\Omega) e^{-jm_0\Omega}}$$

2° Freq shift

$$e^{j\Omega_0 n} x[n] \longleftrightarrow X(\Omega - \Omega_0)$$

proof

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{j\Omega_0 n} e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j(\Omega-\Omega_0)n}$$

$$\boxed{Y(\Omega) = X(\Omega - \Omega_0)}$$

3° Parseval's thm $\sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\Omega)|^2 d\Omega$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

$$x^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\Omega)\, e^{-j\Omega n}\, d\Omega$$

$$x[n]\, x^*[n] = \frac{1}{(2\pi)^2}\iint_{-\pi}^{\pi} X(a)\, X^*(b)\, e^{jan}\, e^{-jbn}\, da\, db$$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{(2\pi)^2}\iint_{-\pi}^{\pi} X(a)\, X^*(b) \underbrace{\left[e^{j(a-b)n} \right]}_{\sum_{n=-\infty}^{\infty}} da\, db$$

$$\boxed{\sum_{n=-\infty}^{\infty} e^{j(a-b)n} = 2\pi\, \delta(a-b)}$$

dirac delta !

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(a)\, X^*(a)\, da$$

4° **Convolution**

$$x[n] \longleftrightarrow X(\Omega)$$

$$h[n] \longleftrightarrow H(\Omega)$$

$$h[n] * x[n] \longleftrightarrow X(\Omega)\, H(\Omega)$$

proof:

$$\sum_{n=-\infty}^{\infty} \left[\sum_{m=-\infty} h[m]\, x[n-m] \right] e^{-j\Omega n}$$

$$= \sum_{m=-\infty}^{\infty} h[m] \left[\underbrace{\sum_{n=-\infty}^{\infty} x[n-m]\, e^{-j\Omega n} e^{j\Omega m}}_{X(\Omega)} \right] e^{-j\Omega m}$$

$$= \sum_{m=-\infty}^{\infty} h[m]\, X(\Omega)\, e^{-j\Omega m} =$$

$$= H(\Omega)\, X(\Omega)$$

Modulation

$$x_1[n]\,x_2[n] \longleftrightarrow \int X_1(\theta)\,X_2(\Omega-\theta)\,d\theta$$

$$x_1[n]\,x_2[n] = \frac{1}{(2\pi)^2}\iint X_1(a)\,X_2(b)\,e^{j(a+b)n}\,da\,db$$

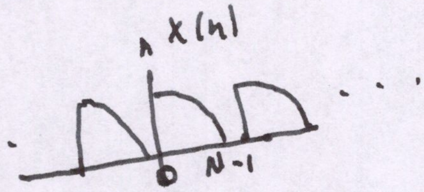
$$\beta = a+b \;=\; b = \beta - a$$

$$db = d\beta$$

$$\frac{1}{(2\pi)^2}\iint X_1(a)\,X_2(\beta-a)\,e^{j\beta n}\,d\beta\,da$$

$$x_1[n]\,x_2[n] = \frac{1}{(2\pi)^2}\int \Big[\underbrace{\int X_1(a)\,X_2(\beta-a)\,da}_{\checkmark}\Big]\,e^{j\beta n}\,d\beta$$

Discrete-time Fourier Series



$$x[n] = x[n - kN]$$

period = N

$$x[n] = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}nk}$$

where

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}$$

proof

$$x[n] = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}nk}$$

$$e^{-j\frac{2\pi}{N}nq} x[n] = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}n(k-q)}$$

$$\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nq} = \sum_{k=0}^{N-1} a_k \Bigg[\underbrace{\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}n(k-q)}}_{N\,\delta[k-q] = \begin{cases} N & k-q=0 \\ 0 & \text{otherwise} \end{cases}} \Bigg]$$

$$\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk} = a_k N$$

$$\boxed{a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}}$$

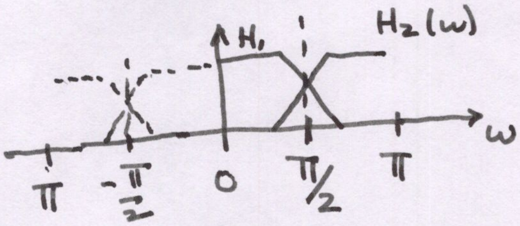
Discrete Fourier Transform (DFT)

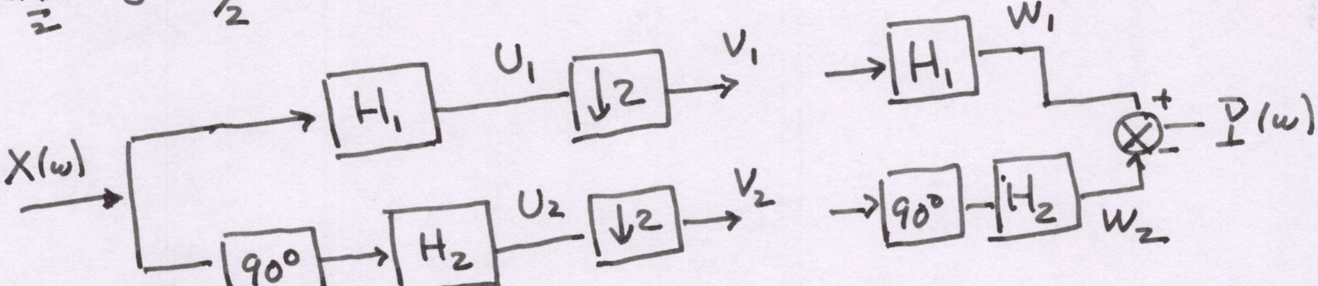
let ~~a_k~~

$$X(k) \equiv \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk} = a_k N$$

$$x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}nk}$$

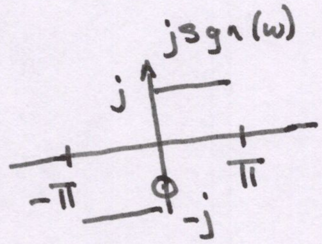
QMF (Quadrature Mirror Filter)





stage 1

$\boxed{90°} \rightarrow j\,\mathrm{sgn}(\omega)$



$-\pi < \omega < \pi$

$$U_1 = H_1 X$$

$$U_2 = j\,\mathrm{sgn}(\omega)\, H_2 X$$

stage 2 Decimation

$$V_1 = U_1 * \sum_{k=-\infty}^{\infty} \delta(\omega - k\pi)$$

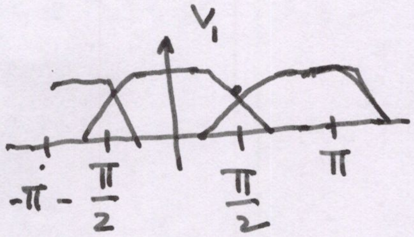
$$V_2 = U_2 * \sum_{k=-\infty}^{\infty} \delta(\omega - k\pi)$$

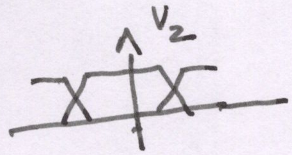
retain image
$-\pi < \omega < \pi$

$$V_1 = U_1(\omega) + U_1(\omega - \pi) + U_1(\omega + \pi)$$

$$V_2 = U_2(\omega) + \underbrace{U_2(\omega - \pi) + U_2(\omega + \pi)}_{\text{aliased}}$$

ii





stage3 reconstruct

$$W_1(\omega) = V_1(\omega)\, H_1$$

$$W_2(\omega) = j\,\mathrm{sgn}(\omega)\, V_2\, H_2$$

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expand

$$V_1 = U_1(\omega) + U_1(\omega-\pi) + U_1(\omega+\pi)$$

$$= X(\omega)H_1 + \bar{X}(\omega-\pi)H_1(\omega-\pi) + \bar{X}(\omega+\pi)H_1(\omega+\pi)$$

$$\Rightarrow V_1 = X(\omega)H_1 + \bar{X}(\omega-\pi)H_1(\omega-\pi) + \bar{X}(\omega+\pi)H_1(\omega+\pi)$$

$$\Rightarrow V_2 = j\,\mathrm{sgn}(\omega)H_2(\omega)\bar{X}(\omega) + j\,\mathrm{sgn}(\omega+\pi)H_2(\omega+\pi)\bar{X}(\omega+\pi) + j\,\mathrm{sgn}(\omega-\pi)H_2(\omega-\pi)\bar{X}(\omega-\pi)$$

$$\boxed{W_1 = V_1 H_1 = X H_1^2 + X(\omega-\pi) H_1(\omega-\pi) H_1 + X(\omega+\pi) H_1(\omega+\pi) H_1}$$

$$W_2 = V_2 \, j\,\mathrm{sgn}(\omega) H_2$$

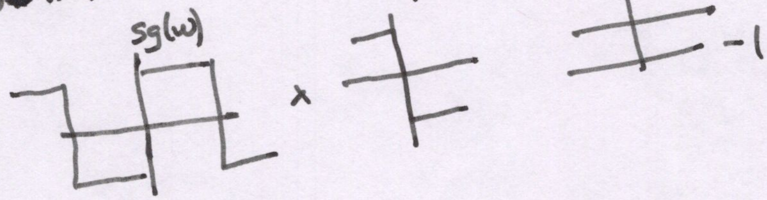
$$= [j\,\mathrm{sgn}(\omega)]^2 H_2^2 X$$

$$+ [j\,\mathrm{sgn}(\omega)][j\,\mathrm{sgn}(\omega+\pi)] H_2(\omega+\pi) H_2(\omega) X(\omega+\pi)$$

$$+ [j\,\mathrm{sgn}(\omega)][j\,\mathrm{sgn}(\omega-\pi)] H_2(\omega-\pi) H_2(\omega) X(\omega-\pi)$$

note $\left.\begin{array}{l}\mathrm{sgn}(\omega)\mathrm{sgn}(\omega+\pi) \\ \mathrm{sgn}(\omega)\mathrm{sgn}(\omega-\pi)\end{array}\right\} = -1$

$\mathrm{sgn}(\omega+\pi)$
$\mathrm{sgn}(\omega-\pi)$



$$\boxed{\begin{aligned} W_2 = & -H_2^2 X \\ & + H_2(\omega+\pi) H_2(\omega) X(\omega+\pi) \\ & + H_2(\omega-\pi) H_2(\omega) X(\omega-\pi) \end{aligned}}$$

$$Y = W_1 - W_2$$

$$= X(\omega)\left[H_1^2 + H_2^2\right]$$

$$+ X(\omega+\pi)\left[H_1(\omega+\pi)H_1(\omega) - H_2(\omega+\pi)H_2(\omega)\right]$$

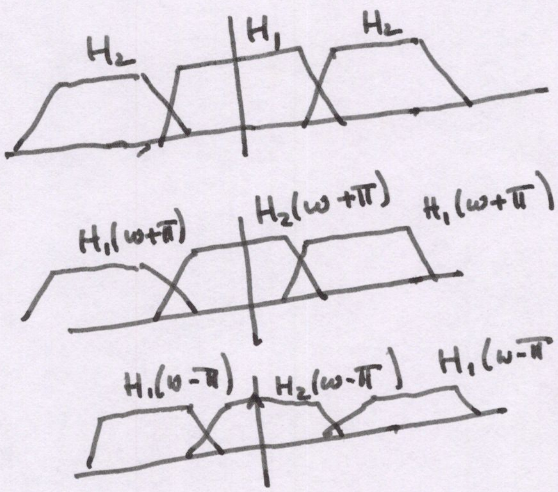
$$+ X(\omega-\pi)\left[H_1(\omega-\pi)H_1(\omega) - H_2(\omega-\pi)H_2(\omega)\right]$$

for perfect reconstruction

$$\boxed{H_1^2 + H_2^2 = 1}$$

$$H_1(\omega+\pi)H_1(\omega) - H_2(\omega+\pi)H_2(\omega) = 0$$

$$H_1(\omega-\pi)H_1(\omega) - H_2(\omega-\pi)H_2(\omega) = 0$$



$$H_1(\omega+\pi) = H_2(\omega)$$

$$H_2(\omega+\pi) = H_2(\omega)$$

~~$H_2(\omega) = H_1$~~

$$H_2(\omega-\pi) = H_1(\omega) \checkmark$$

$$H_1(\omega-\pi) = H_2(\omega)$$

no aliasing obs

$$H_2(\omega \pm \pi) = H_1$$

$$H_1(\omega \pm \pi) = H_2$$

$$\boxed{h_2(n) = (-1)^n h_1(n)}$$