Example

Given

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

prove that

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{+j\omega t}\, d\omega$$

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define

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

$$\text{where } F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt$$

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(a) e^{-j\omega a} da \right] e^{j\omega t}\, d\omega$$

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(a) \underbrace{\left[\int_{-\infty}^{\infty} e^{j\omega(t-a)}\, d\omega \right]}_{2\pi\,\delta(t-a)} da$$

$$y(t) = \frac{2\pi}{2\pi} \int_{-\infty}^{\infty} f(a)\, \delta(t-a)\, da$$

$$\boxed{y(t) = f(t)}$$

Example

Real fonction in time
have conjugate sym. spectru

given $f(t)$ is real

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cos(\omega t)\, dt \; j \int_{-\infty}^{\infty} f(t) \sin(\omega t)\, dt$$

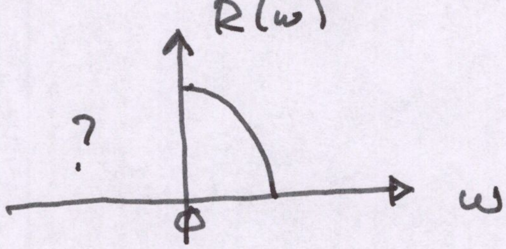
$$F(\omega) = R(\omega) - j\,\underline{X}(\omega)$$

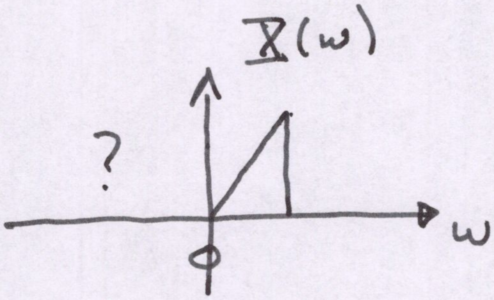
$$R(\omega) = \int_{-\infty}^{\infty} f(t) \cos(\omega t)\, dt$$

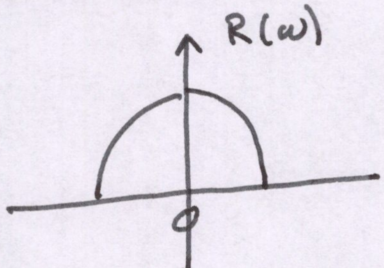
$$\underline{X}(\omega) = \int_{-\infty}^{\infty} f(t) \sin(\omega t)\, dt$$

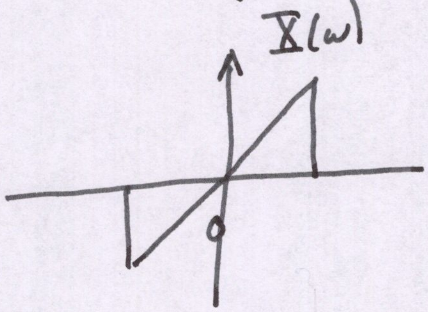
(1) $R(\omega) = R(-\omega)$

(2) $\underline{X}(\omega) = -\underline{X}(-\omega)$


R(ω)
?
ω
0


I(ω)
?
ω
0


R(ω)
0


I(ω)
0

Parseval's Thm

$$\int_{-\infty}^{\infty} |f|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

$f \longleftrightarrow F(\omega)$

$$\Longrightarrow F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$f^* \longleftrightarrow \; ?$

$$F^*(\omega) = \int_{-\infty}^{\infty} f^* e^{j\omega t} dt$$

$$F^*(-\omega) = \int_{-\infty}^{\infty} f^* e^{-j\omega t} dt$$

$$f^* \longleftrightarrow F^*(-\omega) \Longleftarrow$$

$$f f^* \longleftrightarrow \frac{F(\omega) * F^*(-\omega)}{2\pi}$$

$$\int_{-\infty}^{\infty} |f|^2 e^{-j\omega t} dt = \int_{-\infty}^{\infty} \frac{F(z) F^*[-(\omega - z)]}{2\pi} dz$$

$\omega \equiv 0$

$$\int_{-\infty}^{\infty} |f|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(z) F^*(z) dz$$

$$\Longrightarrow \int_{-\infty}^{\infty} |f|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(z)|^2 dz$$

Correlation

$$R(t) = \int_{-\infty}^{\infty} x(t+\tau)\,h(\tau)\,d\tau$$

$$R(t) = x \circledast h \quad \text{cross-col.}$$

$$\mathcal{F}\big(R(t)\big) = S_{xh}(\omega) \quad \text{cross-spectrum}$$

$$S_{xh}(\omega) = \int_{-\infty}^{\infty} R(t)\,e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(t+\tau)\,h(\tau)\,d\tau\right] e^{-j\omega t}\,dt$$

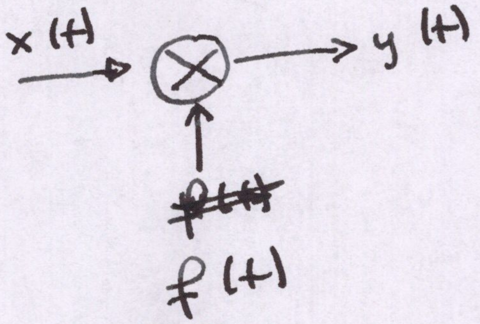
$$= \int_{-\infty}^{\infty} h(\tau)\left[\underbrace{\int_{-\infty}^{\infty} x(t+\tau)\,e^{-j\omega t}\,dt}_{\substack{t+\tau=\alpha \\ dt = d\alpha}}\right] d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\left[\underbrace{\int_{-\infty}^{\infty} x(\alpha)\,e^{-j\omega\alpha}\,d\alpha}_{X(\omega)}\right] e^{j\omega\tau}\,d\tau$$

$$\boxed{S_{xh}(\omega) = X(\omega)\,H(-\omega)}$$

$$\boxed{x \circledast h \longleftrightarrow X(\omega)\,H(-\omega)}$$

Sampling & Reconstruction



where $f(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)\, T_s$

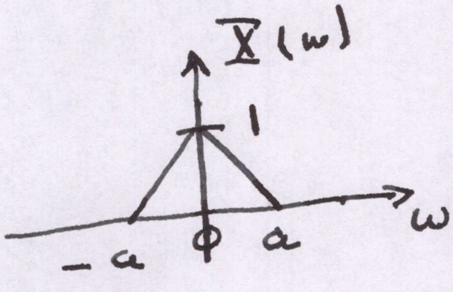
$$FS(f) \Rightarrow f(t) = \sum_{n=-\infty}^{\infty} e^{j\omega_0 n t}$$

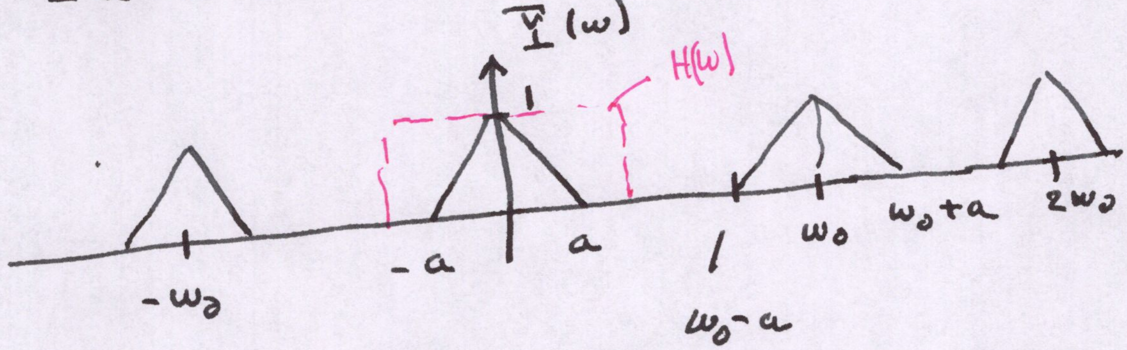
where $\omega_0 = \frac{2\pi}{T_s}$

$$y(t) = x(t) \sum_{n=-\infty}^{\infty} e^{j\omega_0 n t}$$

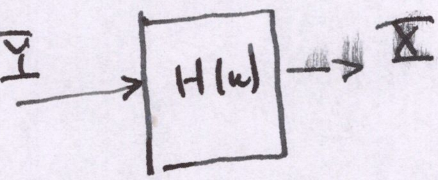
$$Y(\omega) = \frac{X(\omega) * \sum_{n=-\infty}^{\infty} \delta(\omega - \omega_0 n)\, 2\pi}{2\pi}$$

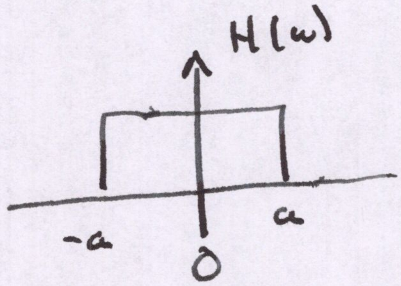
$$\boxed{Y(\omega) = \sum_{n=-\infty}^{\infty} X(\omega - n\omega_0)}$$





<u>Reconstructio</u>





ideal Low-pass filter

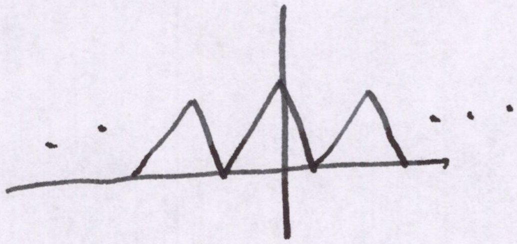
given that

$$\omega_0 - a > a$$

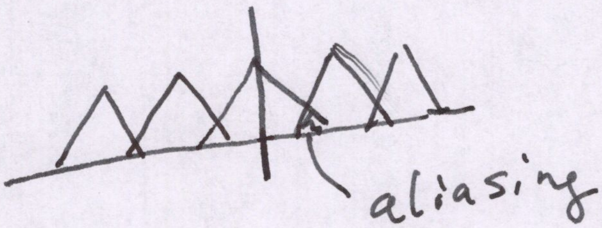
$$\boxed{\omega_0 > 2a}$$

Nyquist freq

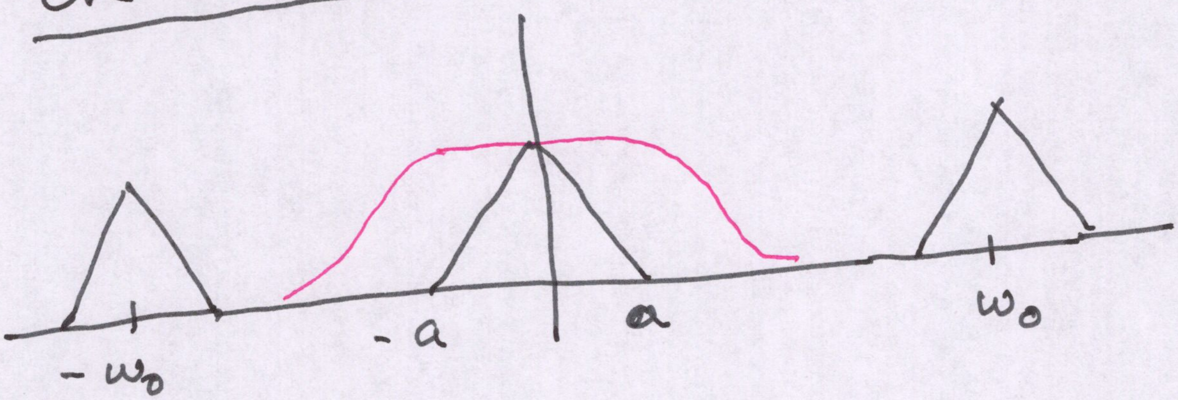
$\omega_0 = 2a$



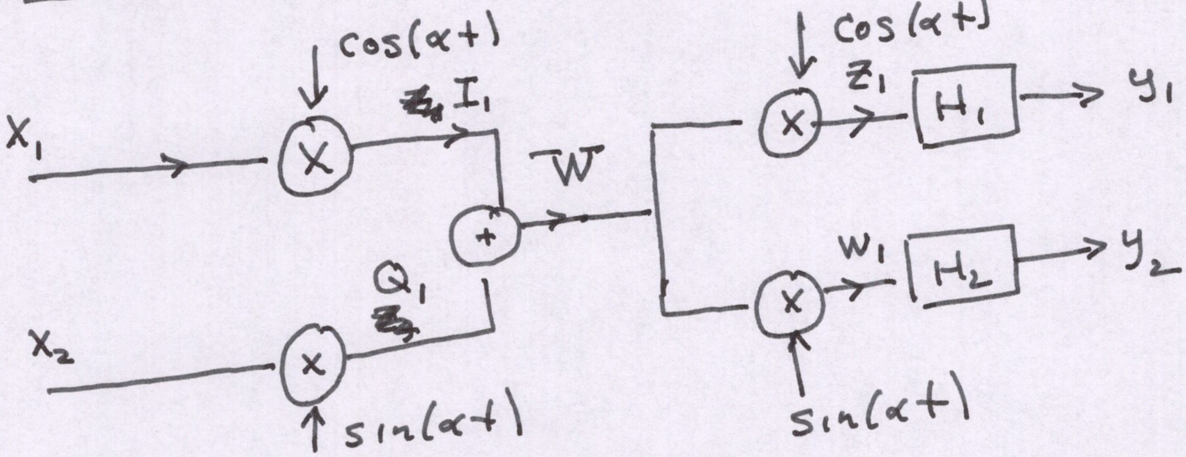
$\omega_0 < 2a$



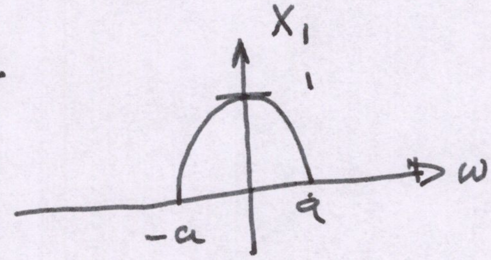
<u>oversampling</u>

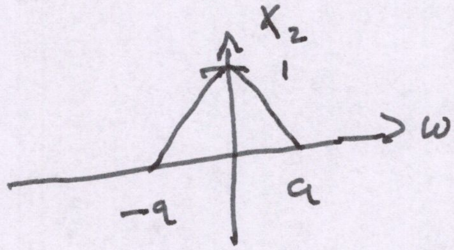


Example



where





① $\mathcal{F}(\cos(\alpha t)) = \pi\left[\delta(\omega-\alpha)+\delta(\omega+\alpha)\right]$

② $\mathcal{F}(\sin(\alpha t)) = -i\pi\left[\delta(\omega-\alpha)-\delta(\omega+\alpha)\right]$

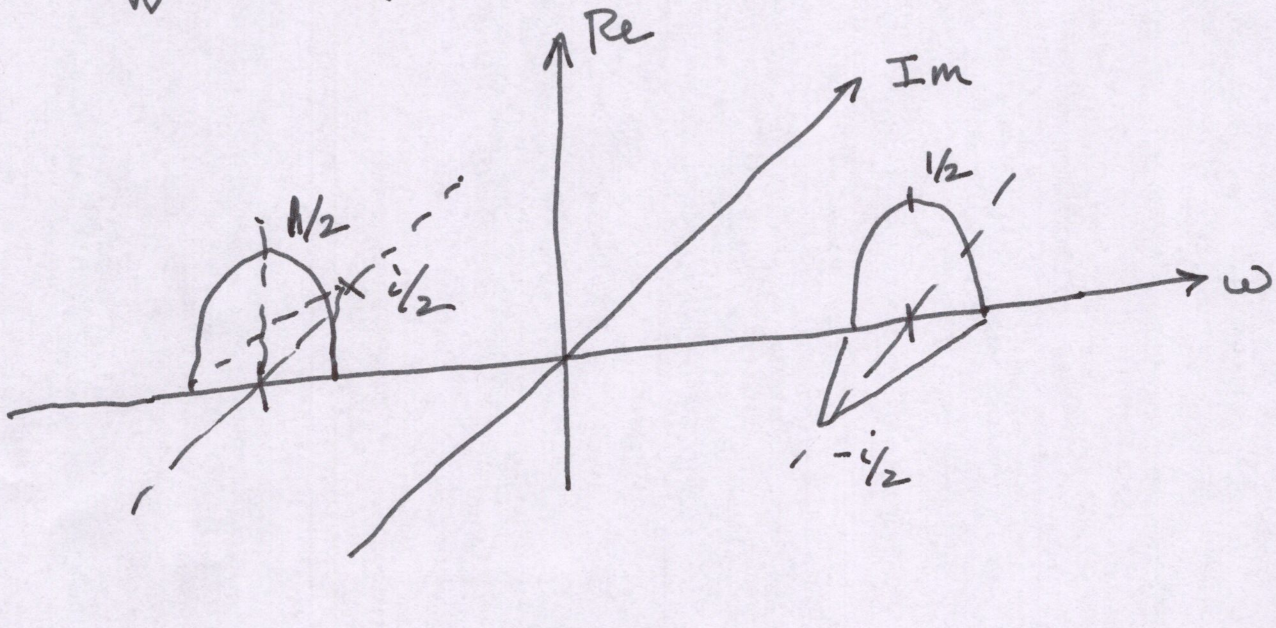
$$I_1 = \frac{1}{2\pi}\,X_1 * \mathcal{F}(\cos(\alpha t))$$

$$Q_1 = \frac{1}{2\pi}\,X_2 * \mathcal{F}(\sin(\alpha t))$$

$$I_1 = \frac{1}{2}\left[X_1(\omega-\alpha)+X_1(\omega+\alpha)\right]$$

$$Q_1 = -\frac{i}{2}\left[X_2(\omega-\alpha)-X_2(\omega+\alpha)\right]$$

$$W = I_1 + Q_1$$



$$Z_1 = \frac{\bar{W}}{2\pi} * \pi\left[\delta(\omega-\alpha) + \delta(\omega+\alpha)\right]$$

$$Z_2 = \frac{\bar{W}}{2\pi} * \pi(-i)\left[\delta(\omega-\alpha) - \delta(\omega+\alpha)\right]$$

$$Z_1 = \frac{1}{2}\left[\bar{W}(\omega-\alpha) + W(\omega+\alpha)\right]$$

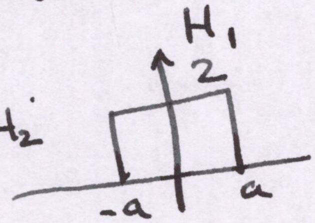
$$Z_2 = \frac{(-i)}{2}\left[W(\omega-\alpha) - \bar{W}(\omega+\alpha)\right]$$

where $W(\omega) = \frac{1}{2}\left[\overline{\underline{X}}_1(\omega-\alpha) + \overline{\underline{X}}_1(\omega+\alpha)\right] - \frac{i}{2}\left[\overline{\underline{X}}_2(\omega-\alpha) - \overline{\underline{X}}_2(\omega+\alpha)\right]$

$$Z_1 = \overline{\underline{X}}_1(\omega)\,\frac{1}{2} + HOT$$

$$Z_2 = \overline{\underline{X}}_2(\omega)\,\frac{1}{2} + H.O.T$$

$H_1 = H_2$



Example

$$y' + 6y = \delta(t)$$

find causal solution

given $y(0) = a$

① $\underline{Y}(\omega) = \int_{0^-}^{\infty} y(t) e^{-j\omega t} dt$

② $\int_{0^-}^{\infty} \frac{dy}{dt} e^{-j\omega t} dt = y e^{-j\omega t} \Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} (-j\omega) y e^{-j\omega t} dt$

$$= -y(0^-) + j\omega \underline{Y}$$

③ $\int_{0^-}^{\infty} \delta(t) e^{-j\omega t} dt = 1$

$$\int_{0^-}^{\infty} y' e^{-j\omega t} dt + \int_{0^-}^{\infty} 6 y e^{-j\omega t} dt = \int_{0^-}^{\infty} \delta(t) e^{-j\omega t} dt$$

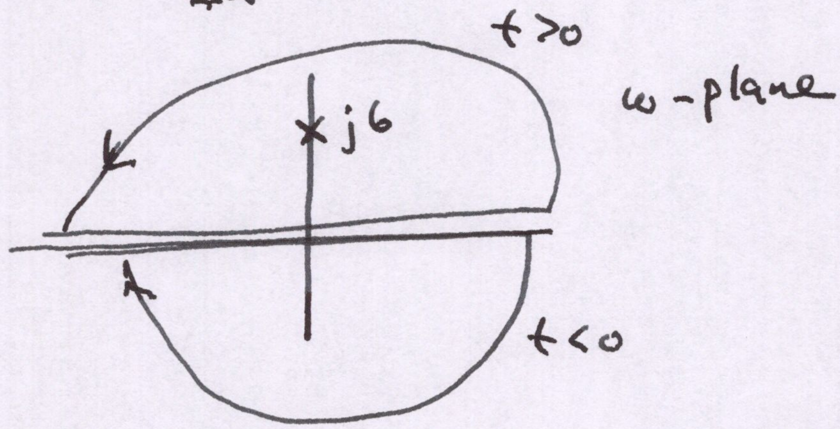
$$-y(0^-) + j\omega \underline{Y} + 6\underline{Y} = 1$$

$$\underline{Y}(j\omega + 6) = 1 + y(0^-)$$

$$\underline{Y} = \frac{1}{j\omega + 6} + \frac{y(0^-)}{j\omega + 6}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{Y}(\omega) e^{j\omega t}\, d\omega = y(t)$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1+y(0^-)}{j(\omega - j6)} e^{j\omega t}\, d\omega$$



$\underline{t>0}$

$$y = \frac{1}{2\pi}\left[\frac{1+y(0^-)}{j(\omega-j6)} e^{j\omega t}\right](\omega - j6)\,2\pi j\,\Big|_{\omega = j6}$$

$$y = e^{-6t}\left(1+y(0^-)\right)$$

$t<0$

$$y = 0$$

$$x(t) \longrightarrow \boxed{H(\omega)} \longrightarrow y(t)$$

$$x(t) = e^{6t} u(-t)$$

$$H(\omega) = \frac{1}{(\omega + j6)(\omega - j6)}$$

find $y(t)$