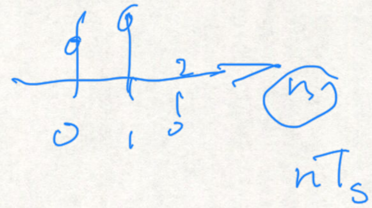
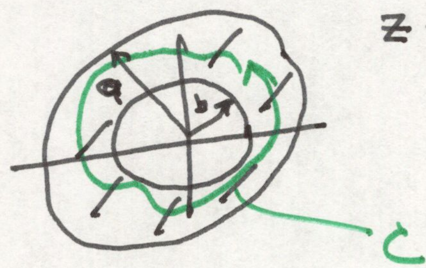
Z-transform



$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \qquad b < |z| < a$$

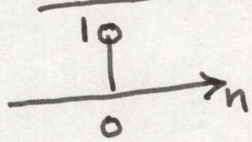
$$x(n) = \frac{1}{2\pi i} \oint_C X(z)\, z^{n-1}\, dz$$

z-plane



Kronecker

The kronocker delta



$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

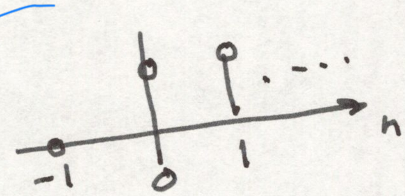
The Dirac delta

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

The unitstep

$n < 0 \quad \mu(n) = 0$

$$\mu[n] = \sum_{k=-\infty}^{n} \delta[k]$$



$$\mu[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Consider

$$x[n] = \alpha^n u[n]$$

note

$$\sum_{n=a}^{b} \alpha^n = \frac{\alpha^a - \alpha^{b+1}}{1-\alpha}$$

find $X(z)$ & ROC

$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n}$$

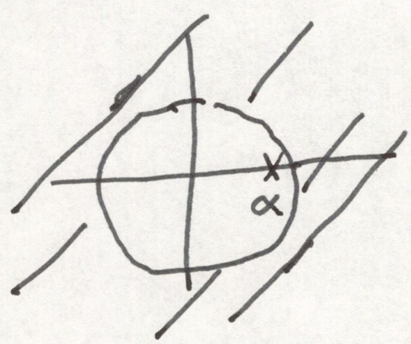
$$" = \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n = \frac{\left(\frac{\alpha}{z}\right)^0 - \left(\frac{\alpha}{z}\right)^\infty}{1 - \frac{\alpha}{z}}$$

for convergence

$$\left|\frac{\alpha}{z}\right| < 1$$

$$\therefore \frac{|\alpha|}{|z|} < 1 \Rightarrow \boxed{|\alpha| < |z|}$$

$$\boxed{X(z) = \frac{1}{1 - \frac{\alpha}{z}}}$$



Consider

$$x[n] = \beta^n u[-n]$$

Find $X(z)$ & ROC

$$X(z) = \sum_{n=-\infty}^{\infty} \beta^n u[-n] z^{-n}$$

$$= \sum_{n=-\infty}^{0} \left(\frac{\beta}{z}\right)^n$$

let $m = -n$

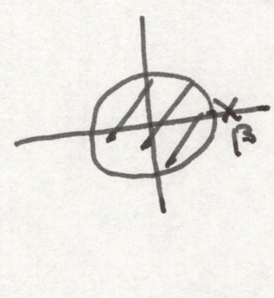
$$= \sum_{m=\infty}^{0} \left(\frac{z}{\beta}\right)^m = \sum_{m=0}^{\infty} \left(\frac{z}{\beta}\right)^m$$

$$= \frac{\left(\frac{z}{\beta}\right)^0 - \left(\frac{z}{\beta}\right)^\infty}{1 - \frac{z}{\beta}}$$

For convergence $\left|\frac{z}{\beta}\right| < 1 \Rightarrow \boxed{|z| < |\beta|}$

$$\boxed{X(z) = \frac{1}{1 - \frac{z}{\beta}}}$$

z-plane



Consider

$$x[n] = \gamma^{|n|}$$

Find $X(z)$ & ROC

$$X(z) = \sum_{n=-\infty}^{\infty} \gamma^{|n|} z^{-n} = \sum_{n=0}^{\infty} \gamma^{n} z^{-n} + \sum_{n=-\infty}^{-1} \gamma^{-n} z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{\gamma}{z}\right)^{n} + \sum_{n=-\infty}^{-1} (\gamma z)^{-n}$$

$$m = -n$$

$$= \sum_{n=0}^{\infty} \left(\frac{\gamma}{z}\right)^{n} + \sum_{m=1}^{\infty} (\gamma z)^{m}$$

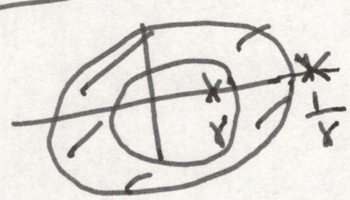
$$= \frac{\left(\frac{\gamma}{z}\right)^{0} - \left(\frac{\gamma}{z}\right)^{\infty}}{1 - \frac{\gamma}{z}} + \frac{(\gamma z)^{1} - (\gamma z)^{\infty}}{1 - \gamma z}$$

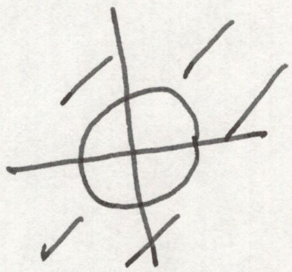
$$\left|\frac{\gamma}{z}\right| < 1 \quad \cap \quad |\gamma z| < 1$$

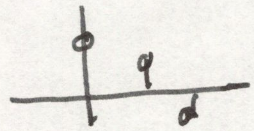
$$|\gamma| < |z| \quad \cap \quad |z| < \frac{1}{|\gamma|}$$

$$\boxed{|\gamma| < |z| < \frac{1}{|\gamma|}}$$

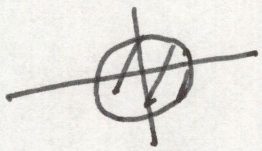
$$\boxed{X(z) = \frac{1}{1 - \frac{\gamma}{z}} + \frac{\gamma z}{1 - \gamma z}}$$

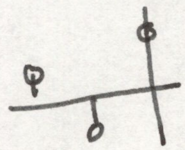




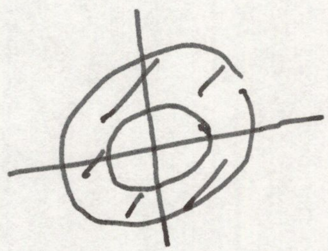


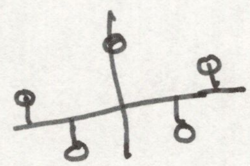
rightsided





leftsided





two-sided

Consider

$$x(n) = \alpha^n n\, u[n]$$

$$X(z) = \sum_{n=0}^{\infty} n \alpha^n z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} n \left(\frac{\alpha}{z}\right)^n$$

note given

$$\Rightarrow f = \sum_{n=0}^{\infty} \gamma^n$$

$$\frac{df}{d\gamma} = \sum_{n=0}^{\infty} n \gamma^{n-1} \Rightarrow \frac{df}{d\gamma} = \frac{1}{\gamma} \sum_{n=0}^{\infty} n \gamma^n$$

$$\therefore \boxed{\gamma \frac{df}{d\gamma} = \sum_{n=0}^{\infty} n \gamma^n \quad \text{where} \quad f = \sum_{n=0}^{\infty} \gamma^n}$$

$$X(z) = \frac{\alpha}{z} \frac{d}{d\left(\frac{\alpha}{z}\right)} \left[\sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n \right]$$

$$= \left(\frac{\alpha}{z}\right) \frac{d}{d\left(\frac{\alpha}{z}\right)} \left[\frac{1 - \left(\frac{\alpha}{z}\right)^{\infty}}{1 - \frac{\alpha}{z}} \right] \qquad \left|\frac{\alpha}{z}\right| < 1$$

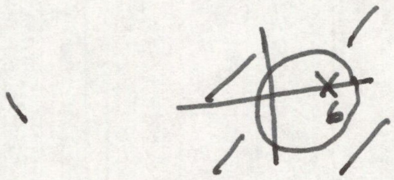
$$\boxed{X = \frac{\alpha}{z} \frac{d}{d\left(\frac{\alpha}{z}\right)} \left[\frac{1}{1 - \frac{\alpha}{z}} \right] \qquad |\alpha| < |z|}$$

Inversion

$$x(n) = \frac{1}{2\pi i}\oint_C X(z)\, z^{n-1}\, dz$$

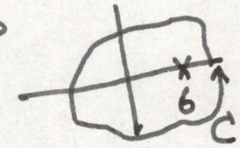
example

$$X(z) = \frac{1}{1 - 6/z} \qquad |z| > 6$$



$$x(n) = \frac{1}{2\pi i}\oint_C \frac{z^n}{z-6}\, dz$$

$n \geq 0$



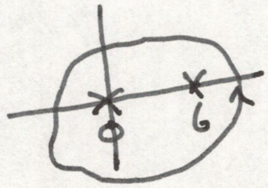
for $n \geq 0$

$$x(n) = \left[\frac{z^n}{z-6}\frac{1}{2\pi i}\right](z-6)\,2\pi i\Big|_{z=6} = 6^n$$

for $n < 0$

say $n = -1$

$$x(-1) = \frac{1}{2\pi i}\oint_C \frac{z^{-1}}{z-6}\, dz$$



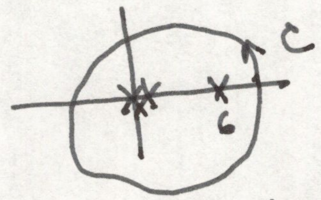
$$x(-1) = \left[\frac{1}{(z-6)(z)\,2\pi i}\right] z\Big|_{z=0} 2\pi i$$

$$+ \left[\frac{1}{(z-6)(z)\,2\pi i}\right](z-6)\,2\pi i\Big|_{z=6}$$

$$x(-1) = \frac{1}{-6} + \frac{1}{6} = 0$$

$n = -2$

$$x(-2) = \frac{1}{2\pi i} \oint_C \frac{z^{-2}}{z-6}\, dz$$



$$x(-2) = \left[\frac{1}{z^2 (z-6)\, 2\pi i} \right] (z-6)\, 2\pi i \Big|_{z=6}$$

$$+ \frac{1}{\cancel{1}!} \frac{d}{dz} \left\{ \left[\frac{1}{z^2 (z-6)\, 2\pi i} \right] z^2\, 2\pi i \right\} \Big|_{z=0}$$

$$x(-2) = \frac{1}{6^2} + \frac{d}{dz}\left(\frac{1}{z-6}\right) \Big|_{z=0}$$

$$x(-2) = \frac{1}{6^2} \ast \frac{1}{(z-6)^2} \Big|_{z=0}$$

$$\boxed{x(-2) = \frac{1}{6^2} - \frac{1}{6^2} = 0}$$

alternative

$\underline{n<0}$

$1/p = z$

$dz = -\frac{1}{p^2}\,dp$

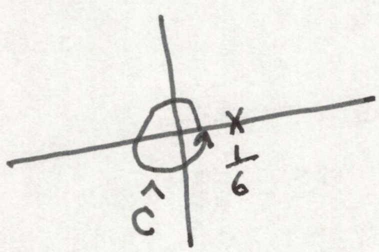
$$\boxed{\begin{array}{c}|z| > 6 \\ |p| < \frac{1}{6}\end{array}}$$

$$x(n) = \frac{1}{2\pi i}\oint_C \frac{z^n}{z-6}\,dz$$

$$= \frac{1}{2\pi i}\oint_{\hat{C}} \frac{1}{p^n\left(\frac{1}{p}-6\right)}\left(-\frac{1}{p^2}\right)dp$$

$$= \frac{1}{2\pi i}\oint_{\hat{C}} \frac{1}{p^{n+1}(1-6p)}\,dp$$

$$= \frac{1}{2\pi i}\oint_{\hat{C}} \frac{1}{p^{n+1}\left(\frac{1}{6}-p\right)6}\,dp$$

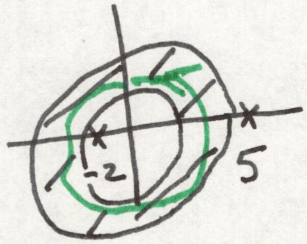


$\underline{n<0}$

$x(n) = 0$

Equivalent

$$X(z) = \frac{1}{(z+2)(z-5)} \qquad 2 < |z| < 5$$

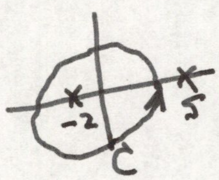


Find $x[n]$

$$x[n] = \frac{1}{2\pi i} \oint_C X(z) z^{n-1}\, dz$$

$$x[n] = \frac{1}{2\pi i} \oint_C \frac{z^{n-1}}{(z+2)(z-5)}\, dz$$

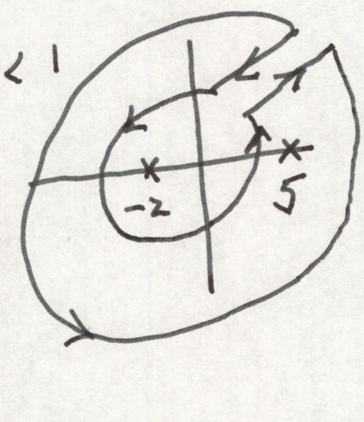
$n \geq 1$



$$x[n] = \left[\frac{z^{n-1}}{2\pi i (z+2)(z-5)} \right] (z+2) 2\pi i \Big|_{z=-2}$$

$$x[n] = \frac{z^{n-1}}{z-5} \Big|_{z=-2}$$

$n < 1$



$$x[n] = \left[\frac{z^{n-1}}{2\pi i (z+2)(z-5)} \right] (z-5)(-2\pi i) \Big|_{z=5}$$

$$x[n] = - \left[\frac{z^{n-1}}{z+2} \right] \Big|_{z=5}$$

Prop

given $x[n] \longleftrightarrow X(z) \quad b_x < |z| < a_x$

$y[n] \longleftrightarrow Y(z) \quad b_y < |z| < a_y$

1. $A\,x[n] + B\,y[n] \longleftrightarrow A X + B Y$

2. $x[n-q] \longleftrightarrow z^{-q} X \quad b_x < |z| < a_x$

3. $x[n] * y[n] \longleftrightarrow X Y \quad b_x < |z| < a_x \cap b_y < |z| < a_y$

4. $x[n] \circledast y[n] \longleftrightarrow X\left(\frac{1}{z}\right) Y(z)$

$b_x < \frac{1}{|z|} < a_x \cap b_y < |z| < a_y$

ie

$b_x < \frac{1}{|z|} \qquad \frac{1}{|z|} < a_x$

$|z| < \frac{1}{b_x} \qquad \frac{1}{a_x} < |z|$

$\left[\frac{1}{a_x} < |z| < \frac{1}{b_x}\right] \cap b_y < |z| < a_y$

Proof

2

$$f = x[n-q]$$

$$F(z) = \sum_{n=-\infty}^{\infty} x[n-q] z^{-n}$$

$$m \equiv n-q$$

$$F(z) = \sum_{m=-\infty}^{\infty} x[m] z^{-m} z^{-q}$$

$$\boxed{F(z) = X(z) z^{-q}} \qquad b_x < |z| < a_x$$

3

$$f(n) = \sum_{k=-\infty}^{\infty} x[k] y[n-k]$$

$$\cancel{= \sum \frac{1}{2\pi}}$$

$$F(z) = \sum_{n=-\infty}^{\infty} \left[\sum_k x[k] y[n-k] \right] z^{-n}$$

$$= \sum_k x[k] \left[\sum_n y[n-k] z^{-n} \right] \quad m = n-k \quad y[m] z^{-m} z^{-k}$$

$$= \sum_k x[k] z^{-k} Y(z) \qquad ROC_x \cap ROC_y$$

$$= X(z) Y(z)$$

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④

$$f(n) = \sum_k x[k+n]\, y[k]$$

$$F(z) = \sum_{n=-\infty}^{\infty} z^{-n} \left[\sum_k x[k+n]\, y(k) \right]$$

$$= \sum_k y(k) \left[\sum_n x[k+n]\, z^{-n} \right]$$

$$m = k+n$$

$$= \sum_k y(k) \left[\sum_m x[m]\, z^{-m} \right] z^{k}$$

$$= Y(z)\, X\left(\frac{1}{z}\right)$$

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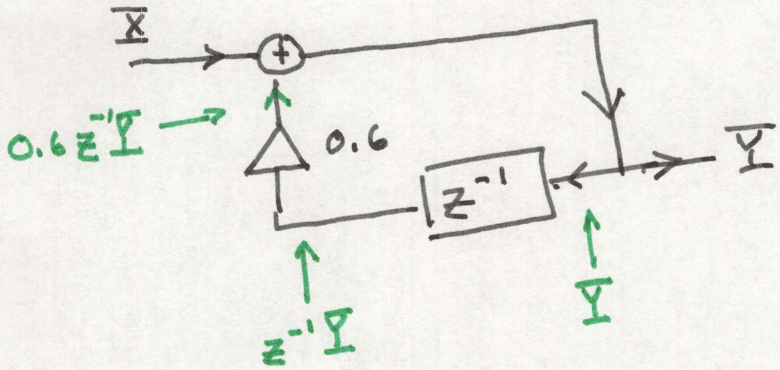
Problem

LTI
causal system



$$Y = Y\,0.6\,z^{-1} + X$$

$$Y\left[1 - 0.6z^{-1}\right] = X$$

$$\boxed{\frac{Y}{X} = \frac{1}{1 - 0.6z^{-1}}}$$

given $\quad x[n] = \delta[n]$

find $\quad y[n] =$

Problem (causal) given $ic = 0$

$$y[n] = -(a+b)\,y[n-1] - ab\,y[n-2] + x[n]$$

(a) Find $\dfrac{Y(z)}{X(z)}$

(b) Find $y[n]$ given $x[n] = \delta[n\cancel{+}-4]$

(a) Find $\dfrac{\underline{Y}(z)}{\underline{X}(z)}$ $\qquad$ $\underline{ic = 0}$

$$\underline{Y} = -(a+b)z^{-1}\underline{Y} - abz^{-2}\underline{Y} + \underline{X}$$

$$\underline{Y}\left[1+(a+b)z^{-1}+abz^{-2}\right] = \underline{X}$$

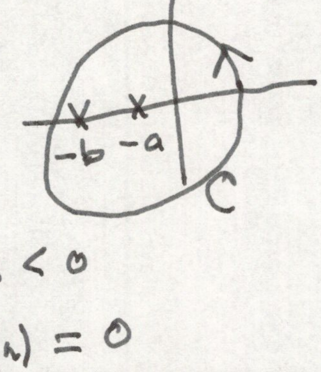
$$\frac{\underline{Y}}{\underline{X}} = \frac{1}{1+(a+b)z^{-1}+abz^{-2}} = \frac{z^2}{(z+a)(z+b)}$$

(b)
$$\left.\begin{aligned} \underline{Y} &= \underline{X}\,\frac{z^2}{(z+a)(z+b)} \\ \underline{X} &= z^{-4} \end{aligned}\right\} \quad \underline{Y} = \frac{z^{-2}}{(z+a)(z+b)}$$

$$y[n] = \frac{1}{2\pi i}\oint_C \frac{z^{n-1}(z^{-2})}{(z+a)(z+b)}\,dz$$

$$y[n] = \frac{1}{2\pi i}\oint_C \frac{z^{n-3}}{(z+a)(z+b)}\,dz$$

$\underline{n-3 \geq 0}$



$n-3 < 0$

$y(n) = 0$