Problems

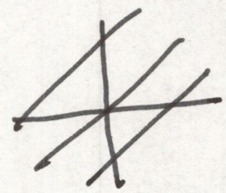
Consider $x(t) = \delta(t)$

Find $X(s)$ & ROC

Recall $X = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

$$X = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt = 1$$

s-plane



$$\boxed{X = 1 \quad ROC_X \text{ is entire}}$$

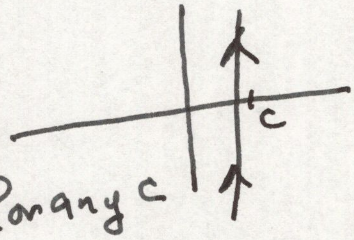
Given $X(s) = 1$ find $x(t)$

$$x(t) = \frac{1}{2\pi i} \int_{-i\infty + c}^{i\infty + c} X e^{st} ds$$

$$x(t) = \frac{1}{2\pi i} \int_{-i\infty + c}^{i\infty + c} 1 \, e^{st} ds$$

CIF cannot be ~~using~~ used

Soln by direct integration



s-plane

for any c

$$\Longrightarrow x(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\,ds$$

<u>Pick c=0</u>

let $\left.\begin{array}{l} s = i\omega \\ ds = i\,d\omega \end{array}\right\} \Longrightarrow x(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{i\omega t}\, i\,d\omega$

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\,d\omega}$$

So:

$$x(t) = \underset{a}{\overset{\infty}{\uparrow}} \frac{1}{2\pi}\int_{-a}^{a} e^{i\omega t}\,d\omega = \underset{a}{\overset{\infty}{\uparrow}} \frac{1}{2\pi}\left.\frac{e^{i\omega t}}{it}\right|_{-a}^{a}$$

$$= \underset{a}{\overset{\infty}{\uparrow}} \frac{1}{2\pi i}\left[\frac{e^{iat} - e^{-iat}}{t}\right]$$

$$= \underset{a}{\overset{\infty}{\uparrow}} \frac{2i}{2\pi i}\,\frac{\sin(at)}{t} = \underset{a}{\overset{\infty}{\uparrow}} \frac{1}{\pi}\,a\left[\frac{\sin at}{at}\right]$$

$$\boxed{x(t) = \underset{a}{\overset{\infty}{\uparrow}} \frac{a}{\pi}\left[\frac{\sin(at)}{at}\right]}$$

prop of

$$x(t) = \overset{\infty}{\underset{a}{\uparrow}} \frac{a}{\pi}\left[\frac{\sin(at)}{at}\right] = \delta(t)$$

1° $x(0) = \infty$ proof: $\overset{\infty}{\underset{a}{\uparrow}} \frac{a}{\pi} = \infty$

2° $\int_{-\infty}^{\infty} x(t)\,dt = 1$

proof: $\overset{\infty}{\underset{a}{\uparrow}} \int_{-\infty}^{\infty} \frac{a}{\pi}\left[\frac{\sin(at)}{at}\right] dt$

let $\theta = at$

$d\theta = a\,dt$

$$\overset{\infty}{\underset{a}{\uparrow}} \int_{-\infty}^{\infty} \frac{1}{\pi}\left[\frac{\sin\theta}{\theta}\right] d\theta = 1$$

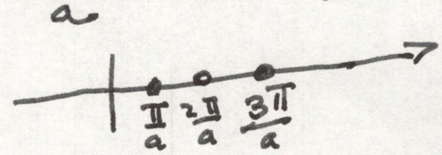
3° $x(t) = 0$ $|t| > 0$

proof: $x(t) \sim \left[\frac{1}{\pi} \overset{\infty}{\underset{a}{\uparrow}} \sin(at)\right] \frac{1}{t}$

density of zeros approach ∞

$at = n\pi$

$\overset{\infty}{\underset{a}{\uparrow}}$ $t = \frac{n\pi}{a} \Rightarrow t = \frac{\pi}{\infty}$



Prop of BLT's

1° Linearity

$$x(t) \longleftrightarrow X(s) \qquad b_x < Re(s) < a_x$$

$$y(t) \longleftrightarrow Y(s) \qquad b_y < Re(s) < a_y$$

$$q(t) = \hat{a}\, x(t) + \hat{b}\, y(t)$$

$$Q(s) = \hat{a}\ X(s) + \hat{b}\ Y(s)$$

ROC_Q: $(b_x < Re(s) < a_x) \cap (b_y < Re(s) < a_y)$

2° Time scaling

given: $x(t) \longleftrightarrow \underline{X}(s)$

$b_x < \mathrm{Re}(s) < a_x$

$x(at) \longleftrightarrow \underline{X}\left(\frac{s}{a}\right)\frac{1}{|a|}$

$a > 0$: $ab_x < \mathrm{Re}(s) < aa_x$

$a < 0$: $aa_x < \mathrm{Re}(s) < ab_x$

proof:

given $\mathbb{X}(s) = \int_{-\infty}^{\infty} x(t) e^{-st} \, dt \quad b_x < Re(s) < a_x$

Find $\mathcal{L}(x(at))$

$\underline{a > 0}$ $\quad \mathcal{L}(x(at)) = \int_{-\infty}^{\infty} x(at) e^{-st} \, dt$

let $\left.\begin{array}{l} \theta = at \\ d\theta = a\,dt \end{array}\right\} \mathcal{L}(x(at)) = \int_{-\infty}^{\infty} \frac{x(\theta) e^{-\frac{s}{a}\theta}}{a} \, d\theta$

$$\therefore \boxed{\mathcal{L}(x(at)) = \frac{1}{a} \mathbb{X}\left(\frac{s}{a}\right)}$$

$$b_x < Re\left(\frac{s}{a}\right) < a_x \Rightarrow \boxed{a\, b_x < Re(s) < a_x a}$$

$\underline{a < 0}$ $\quad \mathcal{L}(x(at)) = \int_{+\infty}^{-\infty} \frac{x(\theta) e^{-\frac{s}{a}\theta}}{a} \, d\theta$

$$\mathcal{L}(x(at)) = \left(-\frac{1}{a}\right) \int_{-\infty}^{\infty} x(\theta) e^{-\frac{s}{a}\theta} \, d\theta$$

$$\boxed{\text{''} = -\frac{1}{a} \mathbb{X}\left(\frac{s}{a}\right)}$$

$$b_x < Re\left(-\frac{s}{|a|}\right) < a_x$$

$$(-|a|) b_x > Re(s) > a_x (-|a|)$$

$$\boxed{a\, b_x > Re(s) > a_x a}$$

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Convolution

given $\quad x(t) \longleftrightarrow X(s) \qquad b_x < Re(s) < a_x$

$\quad h(t) \longleftrightarrow H(s) \qquad b_h < Re(s) < a_h$

$$x * h \longleftrightarrow X H$$

$$[b_x < Re(s) < a_x] \cap [b_h < Re(s) < a_h]$$

proof

$$x * h = \int_{-\infty}^{\infty} x(q)\, h(t-q)\, dq$$

$$\mathcal{L}(x*h) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(q)\, h(t-q)\, dq \right] e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} x(q) \left[\underbrace{\int_{-\infty}^{\infty} h(t-q)\, e^{-st}\, dt}_{\substack{\theta = t-q \\ d\theta = dt}} \right] dq$$

$$= \int_{-\infty}^{\infty} x(q) \left[\underbrace{\int_{-\infty}^{\infty} h(\theta)\, e^{-s\theta}\, d\theta}_{H(s)} \right] e^{-sq}\, dq$$

$$= \left[\int_{-\infty}^{\infty} x(q)\, e^{-sq}\, dq \right] H(s)$$

$$\boxed{\mathcal{L}(x*h) = X(s)\, H(s)}$$

Correlation

given $x(t) \Longleftrightarrow X(s) \quad b_x < Re(s) < a_x$

$y(t) \Longleftrightarrow Y(s) \quad b_y < Re(s) < a_y$

$$\cancel{x(\circledast)}$$

$$x \circledast h \equiv \int_{-\infty}^{\infty} x(q)\, y(t+q)\, dq$$

Find

$$\mathcal{L}(x \circledast h) = Y(s)\, X(-s)$$

$$[b_y < Re(s) < a_y] \cap [b_x < Re(-s) < a_x]$$

$$\cap [-b_x > Re(s) > -a_x]$$

"

proof

$$\mathcal{L}(x \circledast h) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(q)\, y(t+q)\, dq \right] e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} x(q) \left[\int_{-\infty}^{\infty} y(t+q)\, e^{-st}\, dt \right] dq$$

$$\theta = t+q$$

$$d\theta = dt$$

$$= \int_{-\infty}^{\infty} x(q) \left[\underbrace{\int_{-\infty}^{\infty} y(\theta)\, e^{-s\theta}\, d\theta}_{Y(s)} \right] e^{sq}\, dq$$

$$= \int_{-\infty}^{\infty} x(q)\, e^{sq}\, dq\; Y(s)$$

$$\boxed{\mathcal{L}(\;) = X(-s)\, Y(s)}$$

$$\text{ROC} \rightarrow \left(b_x < Re(-s) < a_x\right) \cap \left(b_y < Re(s) < a_y\right)$$

$$\left(-b_x > Re(s) > -a_x\right) \cap \left(b_y < Re(s) < a_y\right)$$

problem

Find the causal soln of

$$\frac{dx}{dt} + 6x = e^{-6t} u(t)$$

where $x(0) = 1$

soln

$$\int_{0^-}^{\infty} \frac{dx}{dt} e^{-st} + 6 \int_{0^-}^{\infty} x e^{-st} dt = \int_{0^-}^{\infty} e^{-(s+6)t} dt$$

evaloation of each integral

1° $\int_{0^-}^{\infty} x e^{-st} dt = X(s)$ $\qquad Re(s) > a$

not known but as big as necessary to obtain $X(s)$

2° $\int_{0^-}^{\infty} e^{-(s+6)t} dt = \left. \frac{e^{-(s+6)t}}{-(s+6)} \right|_0^{\infty}$

$$= \frac{1}{s+6} \quad Re(s) > -6$$

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3°

$$\int_{0^-}^{\infty} \frac{dx}{dt} e^{-st}\,dt \overset{ibp}{=} x e^{-st}\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} (-s)\,x e^{-st}\,dt$$

$$= \cancel{x}\Big[-x(0^-) + \underbrace{x(\infty)e^{-s\infty}}_{0}\Big] + s\,\underline{X}$$

for $\mathrm{Re}(s) > a$

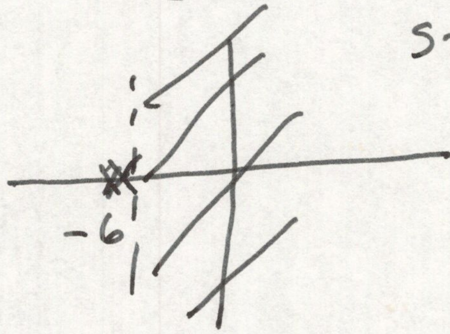
$$s\underline{X} - \cancel{x(0)}^{\,1} + 6\underline{X} = \frac{1}{s+6}$$

$$(s+6)\underline{X} = 1 + \frac{1}{s+6}$$

$$\underline{X} = \frac{1}{s+6} + \frac{1}{(s+6)^2} \qquad \underline{\mathrm{Re}(s) > -6}$$

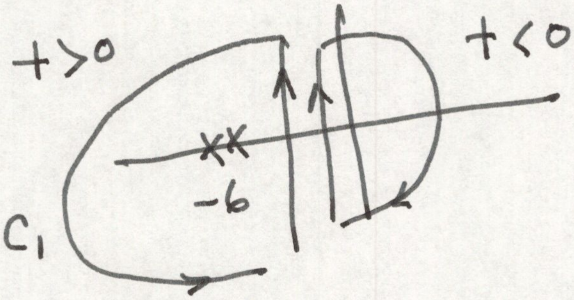
$$\underline{a = -6}$$

$$\underline{X} = \frac{s+7}{(s+6)^2} \qquad \mathrm{Re}(s) > -6$$



s-plane

Inversion



~~t>0~~ $x(t) = 0$ **t<0**

t>0

$$x = \frac{1}{2\pi i}\oint_{C_1} X e^{st}\, ds = \frac{1}{1!}\frac{d}{ds}\left[\frac{X e^{st}}{2\pi i}(s+6)^2\, 2\pi i\right]\Bigg|_{s=-6}$$

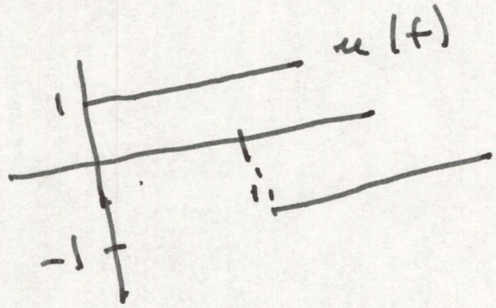
$$= \frac{1}{1!}\frac{d}{ds}\left[(s+7)e^{st}\right]\Bigg|_{s=-6}$$

$$= \frac{1}{1!}\left[e^{st} + (s+7)te^{st}\right]\Bigg|_{s=-6}$$

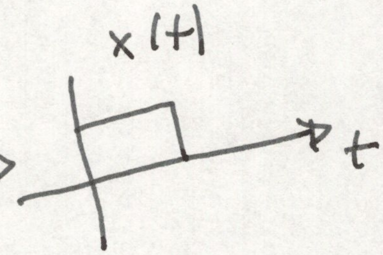
$$\boxed{x = \frac{1}{1!}\left[e^{-6t} + te^{-6t}\right]}$$

Problem

$$x(t) = u(t) - u(t-1)$$



$\Rightarrow$



find

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\,dt = \int_0^1 e^{-st}\,dt$$

$$= \left.\frac{e^{-st}}{-s}\right|_0^1 = -\frac{e^{-s}}{s} + \frac{1}{s}$$

$$X(s) = \frac{1-e^{-s}}{s} \quad \underline{\underline{\text{entire}}}$$

$\therefore$ cannot CIF for inverse BLT

check $\displaystyle \underset{s}{\overset{0}{\uparrow}}\ \frac{1-e^{-s}}{s} \sim \underset{s}{\overset{0}{\uparrow}}\ \frac{+s}{s} = 1$

Better approach

$$X(s) = \underbrace{\int_{-\infty}^{\infty} u(t) e^{-st}\,dt}_{Re(s) > 0} - \underbrace{\int_{-\infty}^{\infty} u(t-1) e^{-st}\,dt}_{Re(s) > 0}$$

$$Re(s) > 0 \ \cap \ Re(s) > 0$$

$$Re(s) > 0$$

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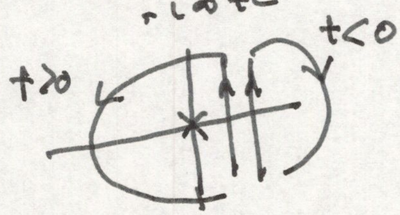
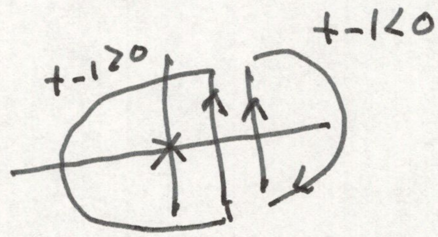
$$\underline{X}(s) = \int_0^\infty e^{-st}\,dt - \int_1^\infty e^{-st}\,dt$$

$$\underline{X}(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$\mathrm{Re}(s) > 0 \cap \mathrm{Re}(s) > 0$$

$$x_1(t) = \frac{1}{2\pi i}\int_{-i\infty+c}^{i\infty+c} \frac{e^{st}}{s}\,ds - \frac{1}{2\pi}\int_{-i\infty+c}^{i\infty+c} \frac{e^{s(t-1)}}{s}\,ds$$





Problem

$x(t) \rightarrow X(s)$ $\quad$ $Re(s) > 6$

$y(t) \Leftrightarrow Y(s)$ $\quad$ $Re(s) < -6$

$x(t) * y(t) \Rightarrow$ exist? $\Rightarrow X(s)\,Y(s)$

$Re(s) > 6 \cap Re(s) < -6$

no intersection

$x * y$ doesnot exist

$x \oplus y \Leftrightarrow X(-s)\,Y(s)$

$$\downarrow$$

$Re(-s) > 6 \cap Re(s) < -6$

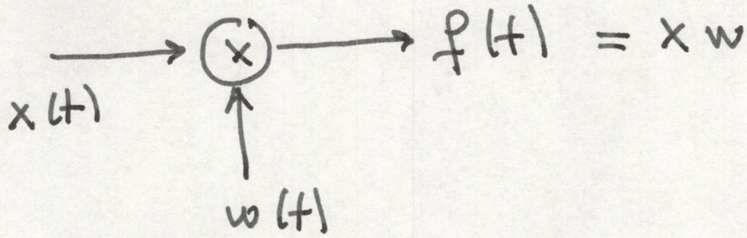
$$\underbrace{Re(s) < -6 \cap Re(s) < -6}_{\text{exist}}$$

If $X(s)$ & $Y(s)$ include the $j\omega$-axis

$\mathcal{L}(x * y)$

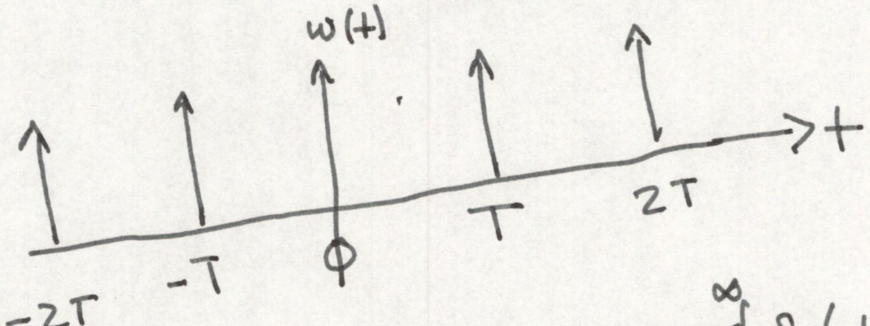
$\mathcal{L}(x \oplus y)$ exists

Consider the system



where $x(t) \longleftrightarrow X(s) \quad b_x < \mathrm{Re}(s) < a_x$

$$w(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$



So $\quad f(t) = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$

Find $F(s)$

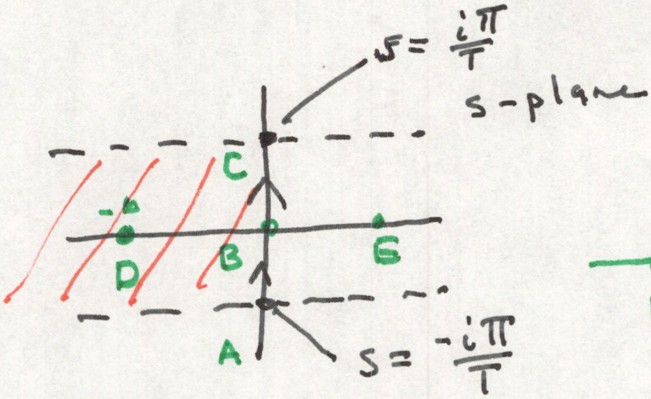
$$F(s) = \int_{-\infty}^{\infty} x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT)\, e^{-st}\, dt$$

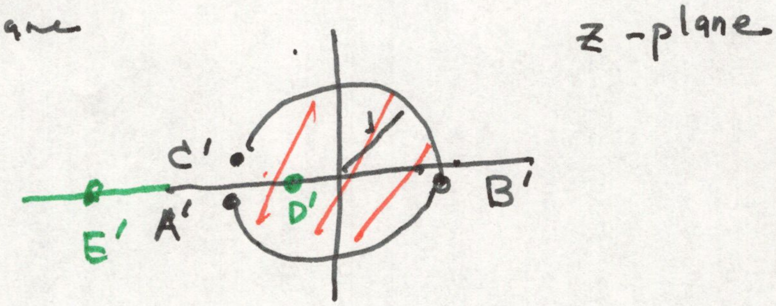
$$= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) e^{-st} \delta(t-kT)\, dt$$

$$\boxed{F(s) = \sum_{k=-\infty}^{\infty} x(kT)\, e^{-ksT}}$$

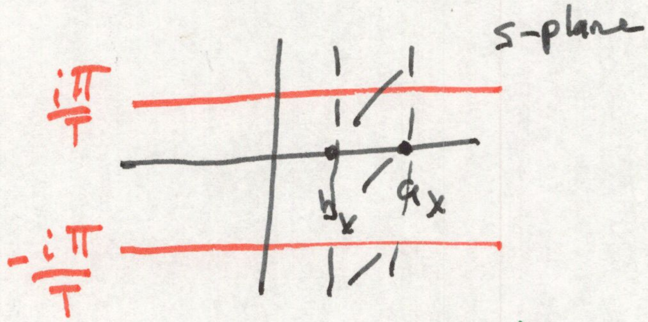
Define

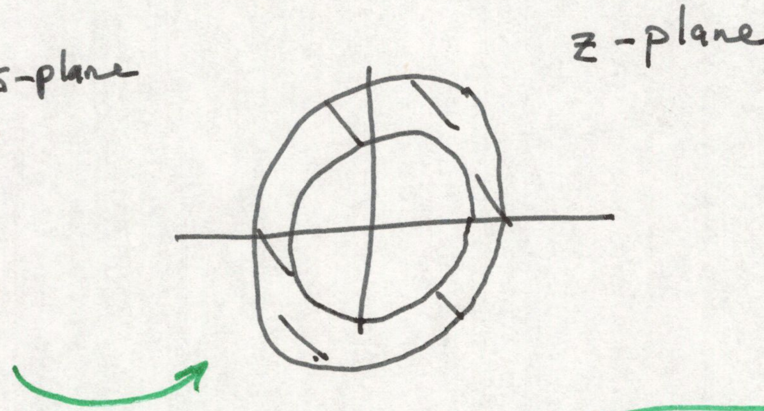
$$\boxed{\begin{aligned} &z = e^{sT} \\ &F(z) = \sum_{k=-\infty}^{\infty} x(kT)\, z^{-k} \qquad z\text{-transform} \end{aligned}}$$





	s	e^{sT}
A	$-\frac{i\pi}{T}$	$-1 = e^{-i\frac{\pi}{T}T}$
B	0	$1 = e^{0}$
C	$\frac{i\pi}{T}$	$-1 = e^{i\frac{\pi}{T}T}$
D	-6	e^{-6T}
E	6	e^{6T}





$b_x < Re(s) < a_x$

Bilinear Trans $$z = e^{sT} = \frac{e^{sT/2}}{e^{-sT/2}} \approx \frac{1+\frac{sT}{2}}{1-\frac{sT}{2}}$$

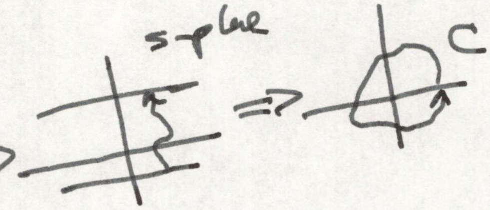
Inverse

$$t = kT$$

$$z = e^{sT} \Longrightarrow \frac{dz}{ds} = Te^{sT} \Rightarrow ds = \frac{1}{T} e^{-sT} dz$$

$$ds = \frac{1}{T} z^{-1} dz$$

$$f(t) = \frac{1}{2\pi i}\int_{-i\infty + c}^{i\infty + c} F(s)\, e^{st}\, ds \Longrightarrow$$

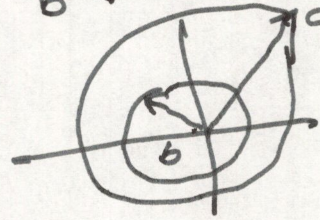


$$\boxed{f(kT) = \frac{1}{2\pi i}\oint_C F(z)\, z^{k}\, z^{-1} \frac{dz}{T}}$$

Therefore given $x(n)$

$$\mathbb{X}(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$b < |z| < a$



$$x(n) = \frac{1}{2\pi i} \oint_{C} \mathbb{X}(z) z^{n-1} \, dz$$