Linear System

$$y = \mathcal{L}(x)$$

Prop of Linear system

1. Homogenity

$$\mathcal{L}(0) = 0$$

2. Scaling of the amplitude

$$\left.\begin{array}{l} \mathcal{L}(x) = y \\ \mathcal{L}(Ax) = Ay \end{array}\right\} \mathcal{L}(Ax) = A\mathcal{L}(x)$$

3. Superposition

$$\mathcal{L}(x_1 + x_2) = \mathcal{L}(x_1) + \mathcal{L}(x_2)$$

problem 1

$$y = mx + b$$

(a) Homogeneity

(b) Amp scaling

(c) superposition

$y = b$

$y = mx + b$

$A(mx+b) = mAx + b$

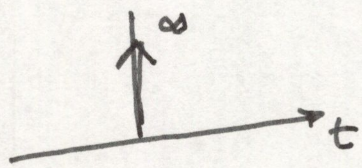
$Amx + Ab \neq mAx + b$

$y = m(x_1 + x_2) + b$

$= mx_1 + b + mx_2 + b$

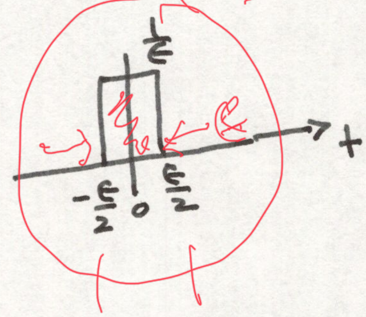
Special functions

1° delta function $\delta(t)$



$\Longleftarrow$ $\epsilon \to 0$

area



props

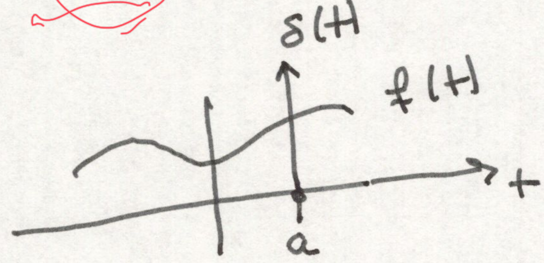
1° $\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$

2° $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

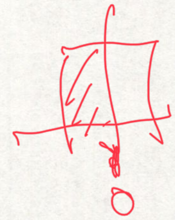
$$\lim_{\epsilon \to 0} \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \frac{1}{\epsilon}\,dt = \lim_{\epsilon \to 0} \frac{t}{\epsilon}\Big|_{-\epsilon/2}^{\epsilon/2} = \lim_{\epsilon \to 0} 1 = 1$$

$t = a$

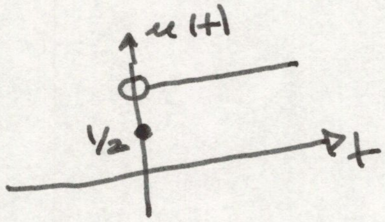
3° $\int_{-\infty}^{\infty} \delta(t-a) f(t)\,dt = f(a)$



2° Step function $u(t)$



$$u(t) = \int_{-\infty}^{t} \cancel{u(q)}\,\delta(q)\, dq = \begin{cases} 1 & t > 0 \\ 1/2 & t = 0 \\ 0 & t < 0 \end{cases}$$



prob:

$$I = \int_{-\infty}^{\infty} \frac{d\delta(q)}{dq} f(q)\, dq$$

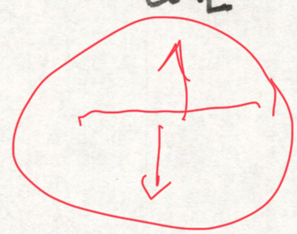
ibp

$$= \delta(q) f(q)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(q) \frac{df(q)}{dq}\, dq$$

$$\boxed{I = 0 - \frac{df(0)}{dq}}$$

$$I(t) = \int_{-\infty}^{\infty} \frac{d\delta(q-t)}{dq} f(q)\, dq$$

$$= \delta(q-t) \frac{df(q)}{dq}\Bigg| - \int \delta(q-t) \frac{df(q)}{dq}\, dq$$



$$\boxed{I(t) = 0 - \frac{df(q)}{dq}}\Bigg|_{q=t}$$

t: clock time

t_o: event

Consider an impulse injected into a ~~linear~~ <u>linear</u> system at $t = t_o$

$$\delta(t - t_o) \longrightarrow \mathcal{L} \longrightarrow y = h(t, t_o)$$

Amp scaling

t clock time

t_o event time

$h(t, t_o)$. impulse response

(a) If the input is $x(t_o)\,\delta(t - t_o)$ what is the output?

$)h(t, t_o)$

$$x(t_o)\,\delta(t - t_o) \longrightarrow \mathcal{L} \longrightarrow y = ?$$

By amp scaling

$$y = x(t_o)\, h(t - t_o)$$

(b) If the input is $\int_{-\infty}^{\infty} x(t_o)\,\delta(t - t_o)\, dt_o$ what is the output?

$$\int_{-\infty}^{\infty} x(t_o)\,\delta(t - t_o)\, dt_o \longrightarrow \mathcal{L} \longrightarrow \int_{-\infty}^{\infty} x(t_o)\, h(t, t_o)\, dt_o$$

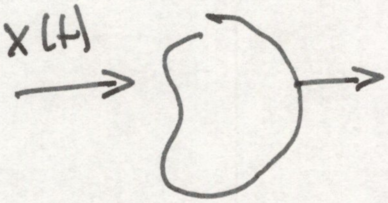
By supperpostion

$$\Downarrow$$

$$x(t)$$

$$\int_{-\infty}^{\infty} x(t_o)\,\delta(t - t_o)\, dt_o = x(t)$$

primary



$$y=\int_{-\infty}^{\infty} x(t_0)\, h(t, t_0)\, dt_0$$

Time invariance

$$h(t, t_0) = h(t - t_0)$$

response is only dependent
on the relative time difference

$$\boxed{y=\int_{-\infty}^{\infty} x(t_0)\, h(t-t_0)\, dt_0}$$

LTI

$$\boxed{y = x * h}$$

$x * h$

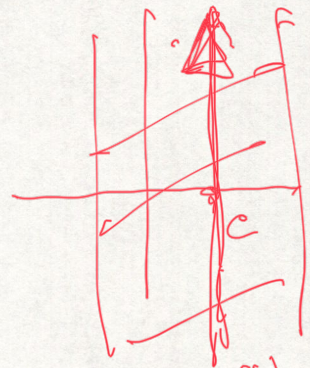
Bilateral Laplace Transform

Given $x(t)$

region of convergence

$$\mathbb{X}(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \qquad \boxed{ROC: b < Re(s) < a}$$

$$x(t) = \frac{1}{2\pi i} \int_{-i\infty + c}^{i\infty + c} \mathbb{X}(s)\, e^{st}\, ds$$



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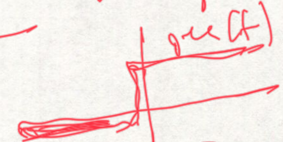
prob

given $x(t) = e^{-6t} u(t)$



Find $\mathbb{X}(s)$ and ROC_x

$$u(t) = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$$

$$\mathbb{X}(s) = \int_{-\infty}^{\infty} e^{-6t} u(t)\, e^{-st}\, dt \equiv \int_0^{\infty} (\quad)\, dt$$

$$\mathbb{X}(s) = \int_0^{\infty} e^{-(s+6)t}\, dt$$

$Re(s+6) > 0$

$$= \left. \frac{e^{-(s+6)t}}{-(s+6)} \right|_0^{\infty} = \frac{e^{-(s+6)\infty}}{-(s+6)} + \frac{e^{-0}}{(s+6)}$$

$\Uparrow$

0

if $Re(s+6) > 0$

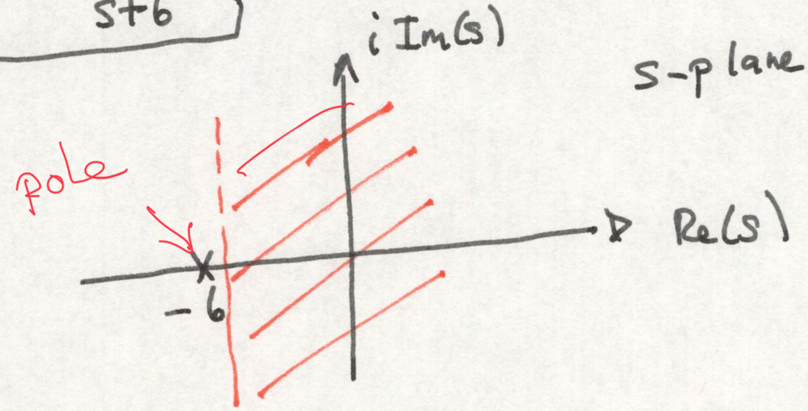
$Re(s+6) > 0$

$Re(s) + Re(6) > 0$

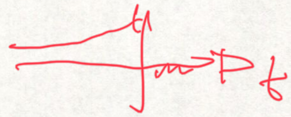
$Re(s) > -Re(6)$ $\quad \boxed{Re(s) > -6}$

$$\boxed{X(s) = \frac{1}{s+6}}$$

$$Re(s+6) > 0 \Rightarrow \boxed{Re(s) > -6}$$



problem



$$x(t) = e^{6t} u(-t)$$

$$X(s) = \int_{-\infty}^{\infty} e^{6t} u(-t) e^{-st}\, dt = \int_{-\infty}^{\infty} e^{-(s-6)t} u(-t)\, dt$$

$$= \int_{-\infty}^{0} e^{-[s-6]t}\, dt$$

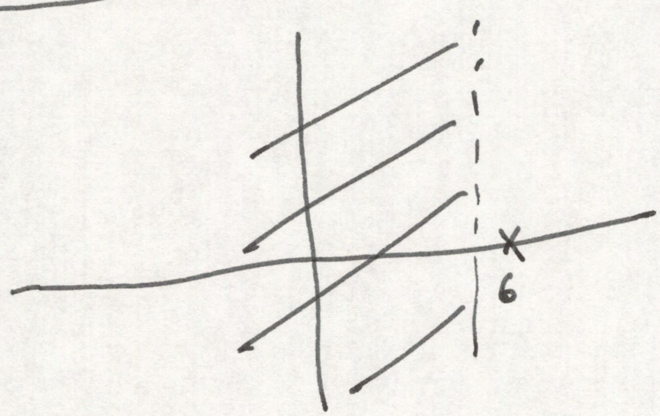
$$= \frac{e^{-[s-6]t}}{-[s-6]}\Bigg|_{-\infty}^{0} = \frac{1}{-[s-6]} + \frac{e^{-[s-6](-\infty)}}{[s+6]}$$

$\to 0$

Ifs $Re(s-6) < 0$

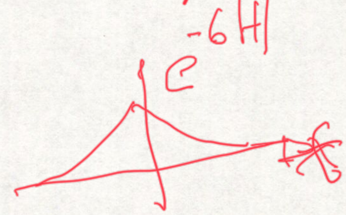
$$\boxed{X(s) = \frac{1}{-[s-6]} \qquad Re(s) < 6}$$

s-plane



$S = 6$

problem



$$x(t) = e^{-6|t|}$$

$$X(s) = \int_{-\infty}^{\infty} e^{-6|t|} e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{-[s-6]t}\, dt + \int_{0}^{\infty} e^{-[s+6]t}\, dt$$

$$= \left. \frac{e^{-[s-6]t}}{-[s-6]} \right|_{-\infty}^{0} + \left. \frac{e^{-[s+6]t}}{-[s+6]} \right|_{0}^{\infty}$$

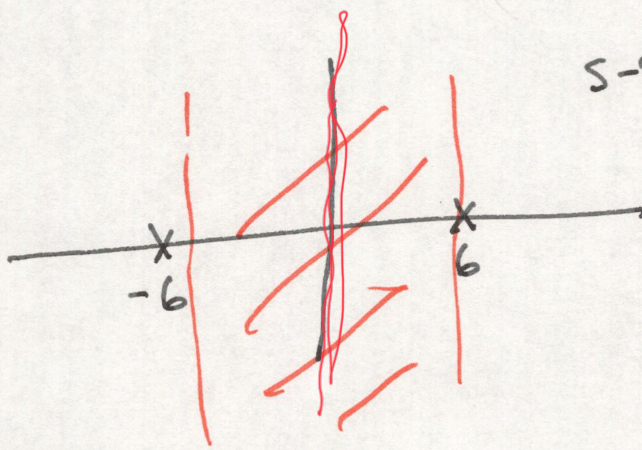
$$\boxed{X(s) = \frac{1}{-[s-6]} + \frac{1}{[s+6]}}$$

$$\mathrm{Re}(s-6) < 0 \cap \mathrm{Re}(s+6) > 0$$

$$\mathrm{Re}(s) < 6 \cap \mathrm{Re}(s) > -6$$

$$\boxed{6 > \mathrm{Re}(s) > -6}$$

s-plane



Summary

$x = e^{-6t}u(t)$

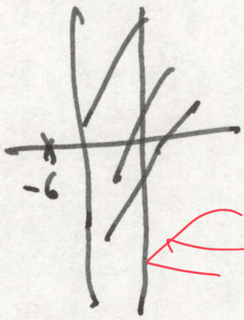
$X = \frac{1}{s+6}$

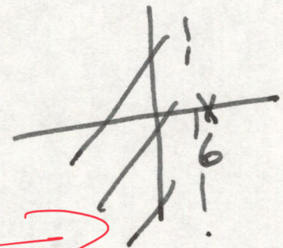
$x = e^{6t}u(-t)$

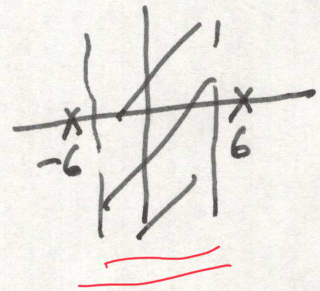
$X = \frac{1}{-[s-6]}$

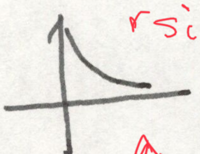
$x(t) = e^{-6t}$

$X = \frac{1}{-[s-6]} + \frac{1}{s+6}$





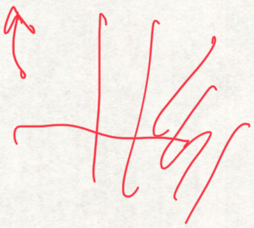


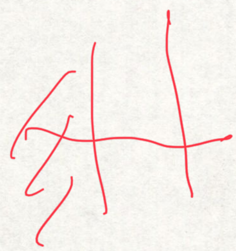


rside

leftside

two sided





ROC

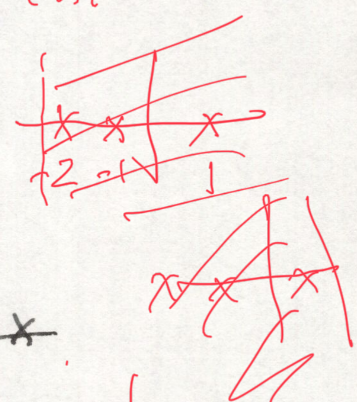
$$X(s) = \frac{1}{(s+1)(s+2)(s-1)}$$

Find

① ROC_x $x(t)$ rightsided causal

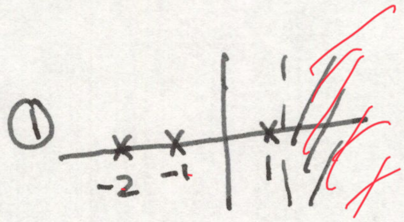
② ROC_x $x(t)$ leftsided anticausal

③ ROC_x $x(t)$ two sided

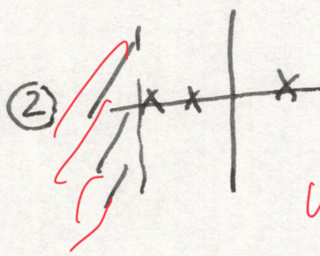


no poles in ROC

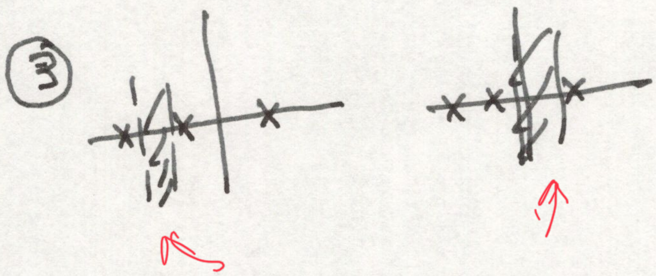
①



②



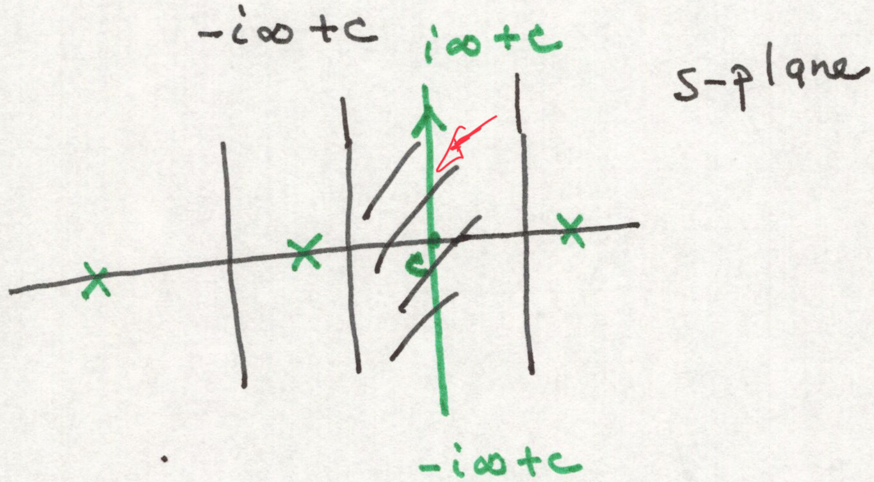
③



Inversion

$$\left.\begin{array}{c} X(s) \\ \text{ROC} \end{array}\right\} \xrightarrow{\mathcal{L}^{-1}} x(t)$$

$$x(t) = \frac{1}{2\pi i} \int_{-i\infty + c}^{i\infty + c} X(s)\, e^{st}\, ds$$

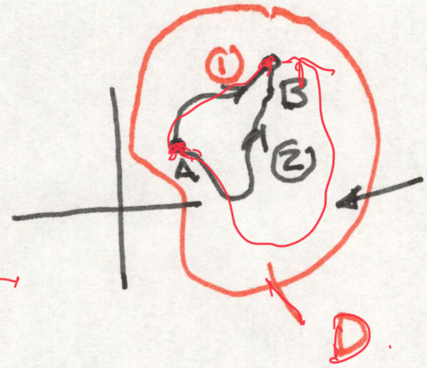


Method Contour integration

Consider

(1) $\underline{X}(s)$

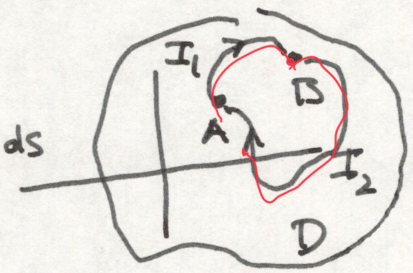
$$I = \begin{cases} \int_A^B \underline{X}(s)\,ds & (1) \\ \int_A^B \underline{X}(s)\,ds & (2) \end{cases}$$



$\underline{X}(s)$ analytic in D

~~$I_{(1)} = I_{(2)}$~~ $\quad I_{(1)} = I_{(2)}$ path independent in D

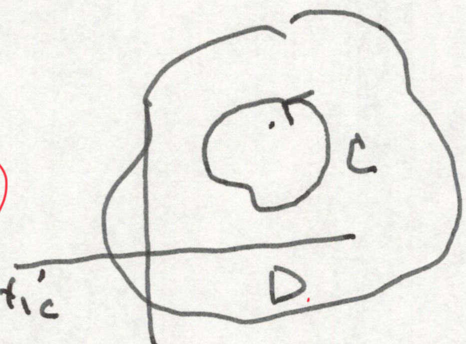
(2) $$Q = \int_A^B \underline{X}(s)\,ds + \int_B^A \underline{X}\,ds$$



if $\underline{X}$ is analytic in D

$$Q = 0$$

(3) $$\oint_C \underline{X}\,ds = 0$$



given $\underline{X}$ is ~~analytc~~ analytic in D

Cauchy–Goursat thm

note Analytic function in D satisfy the Cauchy-Riemann eqn = path indepent

Cauchy - Riemann eqn

$$\Phi(s) = u(s) + i\,v(s)$$

$$s = x + iy$$

$$\Phi(s) = u(x,y) + i\,v(x,y)$$

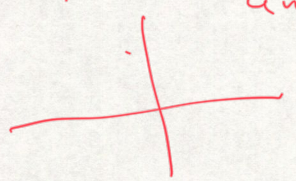
C - R

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$

$\Phi(s) = s \rightarrow$ $\Phi(s)$ is entire

analytic everywhere



$\Phi(s) = s^n \quad n \geq 0 \Rightarrow$ entire

$\Phi(s) = \sum_{n=0}^{N} a_n s^n$ entire

$\Phi(s) = \dfrac{1}{\sum_{n=0}^{N} a_n s^n}$ analytic for $\sum_{n=0}^{N} a_n s^n \neq 0$

ie $\quad \underline{X} = s = x + iy = u + iv$

$u = x$

$v = y$

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{\partial x}{\partial x} - \frac{\partial y}{\partial y} = 0$$

$\underline{X}$ is analytic for all x, y (entire)

① note: $\quad s^n$ for $n \geq 0$ is entire

② $\underline{X}(s) = \sum_{n=0}^{\infty} a_n s^n$ is entire

③ $\quad s^n \quad n < 0$ is analytic for $s \neq 0$

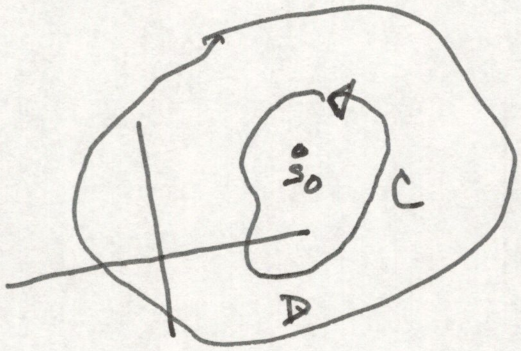
④ $\underline{X}(s) = \dfrac{1}{\sum_{n=0}^{N} a_n s^n}$ is analytic for

$$\sum_{n=0}^{N} a_n s^n \neq 0$$

Cauchy - Integral Eqn

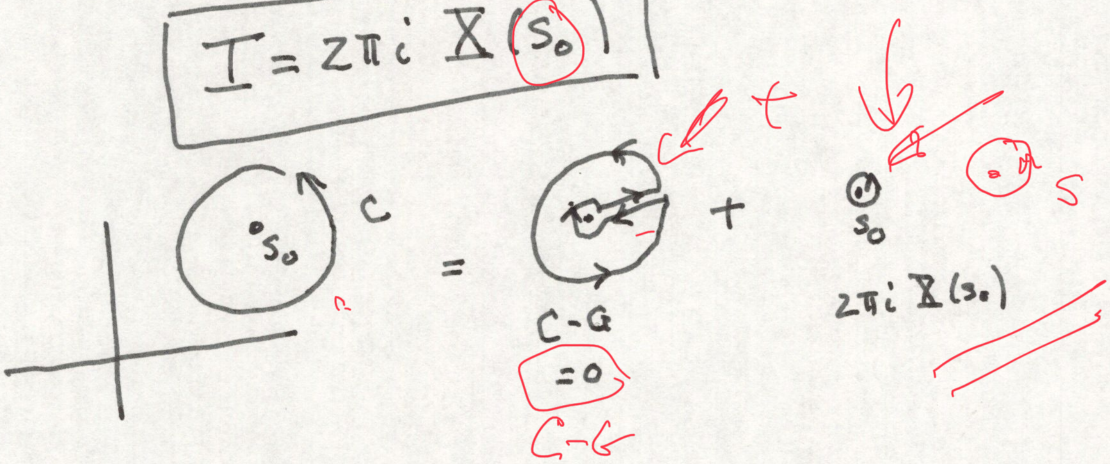
Consider

$$\boxed{I = \oint_C \frac{X(s)}{(s-s_0)}\, ds}$$



$X(s)$ is analytic in D

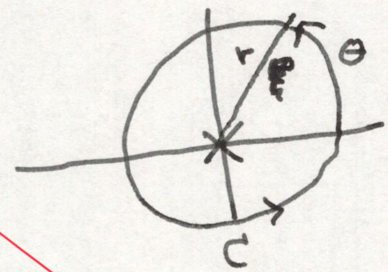
$$\boxed{I = 2\pi i\, X(s_0)}$$



simple example

$\mathbb{X}(s) = \frac{1}{s}$

$I = \oint_C \frac{1}{s}\, ds$



s-plane

$s = re^{i\theta}$

$ds = re^{i\theta}\, i\, d\theta$

$= (1)\, 2\pi i$

$$I = \int_0^{2\pi} \frac{}{\cancel{re^{i\theta}}}\, \cancel{re^{i\theta}}\, i\, d\theta$$

$$I = \int_0^{2\pi} i\, d\theta = i2\pi \checkmark$$

Caughy ~~[illegible]~~ Integral Extensio

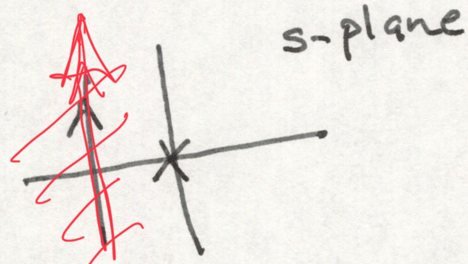
$$I = \oint_C \frac{X(s)}{(s-s_0)^n}\, ds$$

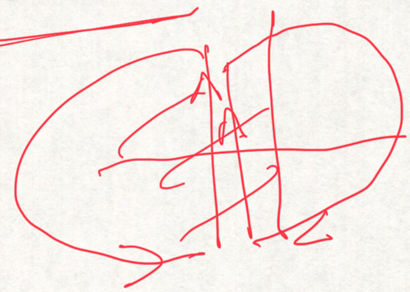
$$= 2\pi i \frac{1}{(n-1)!} \left. \frac{d^{n-1} X(s)}{ds^{n-1}} \right|_{s=s_0}$$

Inversion of the Bi-Laplace transform

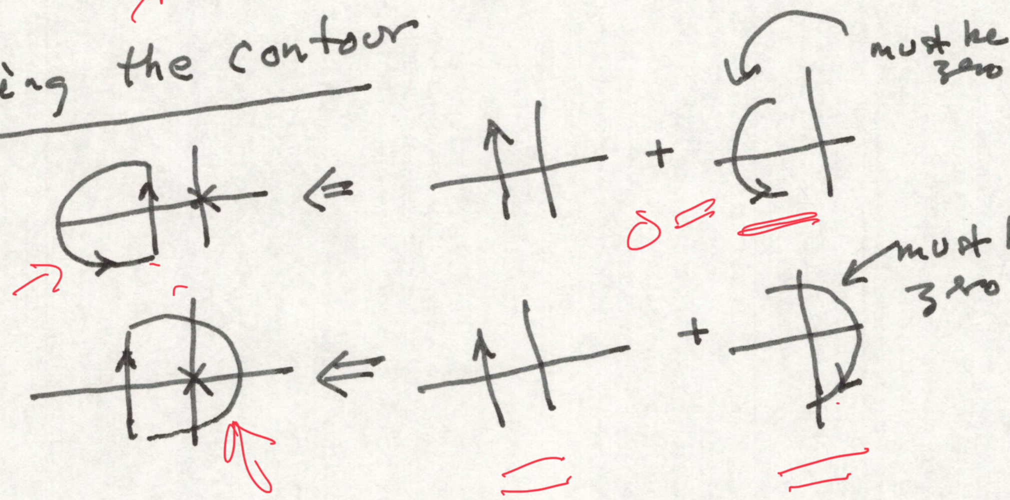
$$x(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} X(s)\, e^{st}\, ds$$

Consider $X(s) = \frac{1}{s}$ $\qquad$ $Re(s) < 0$





Closing the contour

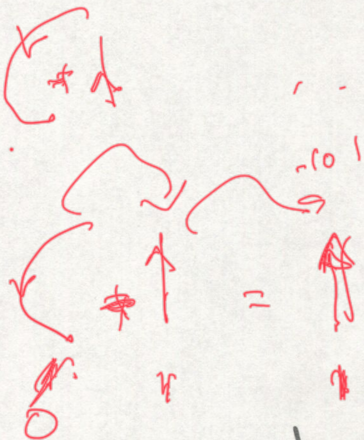


$e^{-10^6(t)} \approx 0$ $\qquad$ $e^{10^6} \approx \infty$

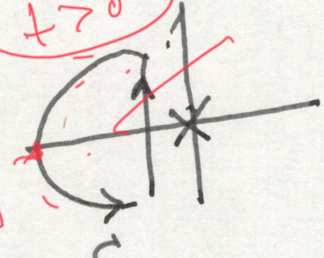
e^{-10}

$$x(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{k}{s} e^{st}\, ds = \frac{1}{2\pi i}\oint_{C_1} \frac{e^{st}}{s}\, ds$$

$e^{-10^6(t)} = 0$



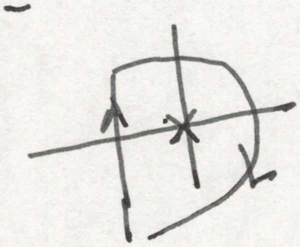
$t > 0$



C_1

$$\boxed{\begin{array}{l} x(t) = 0 \\ \text{if } \ t > 0 \end{array}}$$

$$x(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s} = \frac{1}{2\pi i}\oint_{C_2} \frac{e^{st}}{s}\, ds$$



if $t < 0$

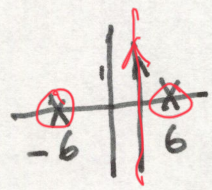
clockwis rot

$$x(t) = 2\pi i \, \frac{1}{2\pi i} \cdot \frac{e^{st}}{s} (2\pi i s) \Big|_{s=0} (-1)$$

$$\boxed{\begin{array}{l} t < 0 \\ x(t) = -1 \end{array}}$$

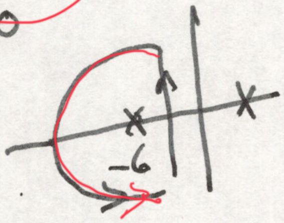
problem

$$X(s) = \frac{1}{(s+6)(s-6)} \qquad -6 < Re(s) < 6$$



$$x(t) = \frac{1}{2\pi i}\int_{-i\infty+c}^{i\infty+c} \frac{e^{st}}{(s+6)(s-6)}\, ds = x(t)$$

$t>0$

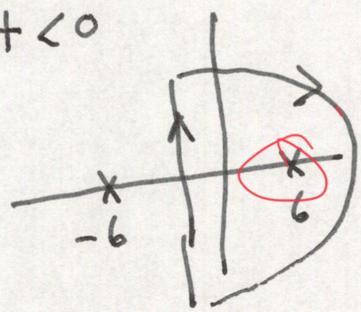


$$x(t) = \left[\frac{1}{2\pi i}\frac{e^{st}}{(s+6)(s-6)}\right](s+6)\,2\pi i\,\Big|_{s=-6} = x(t)$$

counterclockwise

$\oint$

$t<0$



$$x(t) = \left[\frac{1}{2\pi i}\frac{e^{st}}{(s+6)(s-6)}\right](s-6)\,2\pi i\,(-1)\Big|_{s=6}$$

clockwise

$\oint$ $\oint$

<-1

$$x = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} 1 e^{st}\, ds$$

$$s = i\omega \qquad ds = i\, d\omega$$

$$x(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{j\omega t}\, d\omega\, i$$

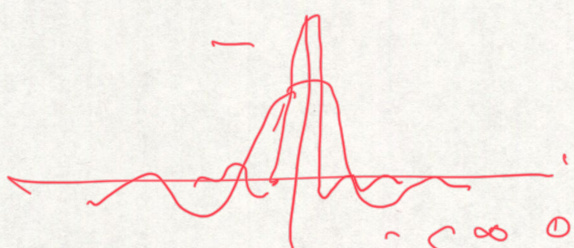
donot do this

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\, d\omega \Rightarrow \frac{1}{2\pi}\,\frac{e^{j\omega t}}{jt}\Big|_{-\infty}^{\infty}$$

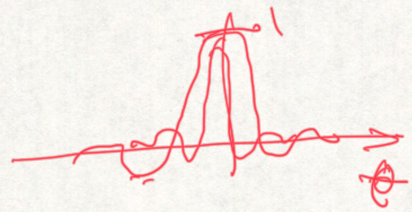
$$= \frac{1}{2\pi}\lim_{\theta\to\infty}\int_{-\theta}^{\theta} e^{j\omega t}\, d\omega = \frac{1}{2\pi}\lim_{\theta\to\infty}\frac{e^{j\omega t}}{jt}\Big|_{\omega=-\theta}^{\theta}$$

$$\frac{1}{2\pi}\left[\frac{e^{j\theta t}}{jt} - \frac{e^{-j\theta t}}{jt}\right]$$

$$= \frac{1}{\pi}\lim_{\theta\to\infty}\frac{\cancel{j}\sin(\theta t)}{\cancel{j}t} = \lim_{\theta\to\infty}\frac{1}{\pi}\left[\frac{\sin\theta t}{\theta t}\right]\theta$$



$$x = \begin{cases}\infty & 0 = t\\ 0 & t \neq 0\end{cases}$$



$x(0) = \infty$

~~$x(t)$~~