energy spectral density to the output energy spectral density. Because of the magnitude-squared nature of these terms, the output and the input energy spectral densities are both independent of any phase variations that might be present.

Another use for Parseval's theorem is in what is called **energy localization.** Assume for some given $f(t)$ that the left-hand side of Equation 8-51 can be computed. This yields the total energy contained in the signal. Now on the right-hand side of Equation 8-51, note first that $|F|^2$ is an even function of ω. Thus we can write:

$$\int_{-\infty}^{\infty} f^2(t)\,dt = \frac{1}{\pi}\int_0^{\infty} |F(j\omega)|^2\,d\omega \qquad (8\text{-}53)$$

Often the energy spectrum $|F|^2$ will be concentrated over a finite band of frequencies. A typical question in this area is to determine such a frequency band within which a certain percentage of the total energy will be localized.

EXAMPLE 8-26

Determine a frequency band $(0, \omega_c)$ over which one half the energy in $f(t) = e^{-t}u(t)$ will be localized.

Solution. The energy in $f(t)$ is:

$$\mathscr{E} = \int_{-\infty}^{\infty} f^2(t)\,dt = \int_0^{\infty} e^{-2t}\,dt = 0.5$$

Now, from Equation 8-53, we can write:

$$\frac{1}{2}(0.5) = \frac{1}{\pi}\int_0^{\omega_c} |F|^2\,d\omega$$

equating one half the energy to the integral with finite upper limit. We know that:

$$F(j\omega) = \frac{1}{1 + j\omega}$$

Thus

$$|F|^2 = \frac{1}{1 + \omega^2}$$

and

$$0.25 = \frac{1}{\pi}\int_0^{\omega_c} \frac{1}{1 + \omega^2}\,d\omega = \frac{1}{\pi}\{\text{Tan}^{-1}\,\omega|_0^{\omega_c}\}$$

or

$$0.25\pi = \text{Tan}^{-1}\,\omega_c - \text{Tan}^{-1}\,0 = \text{Tan}^{-1}\,\omega_c$$

therefore

$$\omega_c = \text{Tan}\,(\pi/4) = 1 \text{ rad/s}$$

The discussion on Parseval's theorem provides a transition between the properties and the applications of the Fourier transform. The result postulated in Parseval's theorem employs the idea of signal energy and follows directly from the definitions of the Fourier transform and the inverse Fourier transform. Using Parseval's theorem in the energy localization problem introduces Fourier transform applications. Applications of the Fourier transform span a wide variety of disciplines. Some of these applications will be dealt with in Section 8.6.

At this point, we pause in order to consolidate our results. We studied the Fourier series and from it developed the Fourier transform. A number of properties of the Fourier transform were considered, not only as an aid to obtain Fourier transform functions, but also as a means to gain deeper insights into the essence of the Fourier transform. Even further appreciation can be obtained by comparing the Fourier transform to the Laplace transform, which has already been discussed in Chapters 4 and 5. A basic understanding of the Laplace transform is presupposed. The next short section deals with the relationship between the Fourier and Laplace transforms.

8-5 THE FOURIER TRANSFORM AND THE LAPLACE TRANSFORM: A COMPARISON

From a cursory glance at the two transforms we might conclude that $F(j\omega)$ is just $F(s)$ with s replaced by $j\omega$. This, however, is not always the case. It is so if $f(t) = 0,\ t < 0$, and $\int_0^{\infty} |f(t)|\,dt < \infty$; that is, if $f(t)$ is absolutely integrable.

EXAMPLE 8-27

Determine $F(j\omega)$ from $F(s)$ for:

(a) $f_1(t) = e^{-10t}u(t)$
(b) $f_2(t) = e^{-t}\,\text{Cos}\,10tu(t)$
(c) $f_3(t) = u(t) - u(t - 10)$

Solution

(a)

$$F_1(s) = \frac{1}{s + 10}$$

Since $f_1(t)$ is zero for $t < 0$ and $f_1(t)$ is absolutely integrable:

$$F_1(j\omega) = \frac{1}{10 + j\omega}$$

(b)

$$F_2(s) = \frac{s + 1}{(s + 1)^2 + 100}\,\cdot\quad F_2(j\omega) = \frac{j\omega + 1}{(j\omega + 1)^2 + 100}$$

(c)

$$F_3(s) = \frac{1}{s} - \frac{1}{s}e^{-10s}\,\cdot\quad F_3(j\omega) = \frac{1}{j\omega} - \frac{1}{j\omega}e^{-10j\omega}$$

Now in this case $u(t) \leftrightarrow F(j\omega) = \pi\delta(\omega) + 1/j\omega$ and the shifted $u(t)$ has the transform:

$$u(t-10) \leftrightarrow e^{-j\omega 10} F(j\omega) = e^{-j\omega 10}\left(\pi\delta(\omega) + \frac{1}{j\omega}\right)$$
$$= e^{0}\,\pi\delta(\omega) + e^{-10j\omega}\frac{1}{j\omega}$$
$$= \pi\delta(\omega) + \frac{e^{-10j\omega}}{j\omega}$$

Therefore
$$F_3(j\omega) = \left(\pi\delta(\omega) + \frac{1}{j\omega}\right) - \left(\pi\delta(\omega) + \frac{e^{-10j\omega}}{j\omega}\right)$$
$$= \frac{1}{j\omega} - \frac{1}{j\omega}e^{-10j\omega}$$

which checks with the preceding result.

Under the two constraints of $f(t) = 0$, $t < 0$, and $f(t)$ being absolutely integrable, we can do the reverse and get $F(s)$ from $F(j\omega)$ by letting $F(s) = F(j\omega)|_{\omega=s/j}$. If we employ the two-sided Laplace transform, we can relax the constraint that $f(t)$ be a causal signal; that is $f(t) = 0$, $t < 0$. The interesting cases, however, are those in which simple substitution does not work. This occurs when absolute integrability does not hold. To handle these cases, assume $f(t)$ is causal so we need only consider the one-sided Laplace transform. If this is the case, then the Laplace transform is more inclusive than the Fourier transform; it exists for a wider class of functions. To put it differently, the existence of $F(j\omega)$ implies the existence of $F(s)$, but the existence of $F(s)$ does not necessarily imply the existence of $F(j\omega)$. Let us examine the issue of absolute integrability by distinguishing the various possible regions of convergence of a given Laplace transform function. These regions can be considered in terms of the s plane pole locations of $F(s)$.

Region of Convergence I If $F(s)$ has all poles in the LHP (Left Hand Plane), then $f(t)$ is absolutely integrable

and $$F(j\omega) = F(s)|s = j\omega \qquad (8\text{-}54)$$

and $$F(s) = F(j\omega)|\omega = s/j \qquad (8\text{-}55)$$

Region of Convergence II If $F(s)$ has any nonrepeated poles on the $j\omega$ axis (with possibly other poles in the LHP), then $f(t)$ is not absolutely integrable but it is a power signal. The Fourier transform of these signals contains impulses in the frequency domain. We can then write:

$$F(j\omega) = F(s)|_{s=j\omega} + \pi\sum_i k_i\delta(\omega - \omega_i) \qquad (8\text{-}56)$$

The k_i terms are the residues at the poles on the $j\omega$ axis: $s = j\omega_i$. The reverse is easier. Given $F(j\omega)$, simply let $\omega = s/j$ and zero out all impulses $\delta(\omega - \omega_i)$ in order to get $F(s)$ from $F(j\omega)$. The case of repeated poles on the $j\omega$ axis is more difficult because $F(j\omega)$ contains $\dot{\delta}$, $\ddot{\delta}$, and so on, terms. To get $F(j\omega)$ in these cases, we obtain $f(t)$ from $F(s)$, then work with $f(t)$ instead of $F(s)$. To get $F(s)$ from $F(j\omega)$ is also easy: Simply let $\omega = s/j$ and zero out all δ, $\dot{\delta}$, $\ddot{\delta}$, and so on, terms.

Region of Convergence III If $F(s)$ has any poles in the RHP (Right Hand Plane), then $F(j\omega)$ does not exist.

EXAMPLE 8-28

(a) Given $F(s) = 10/s(s+10)$, determine $F(j\omega)$.
(b) Given $F(j\omega) = 10/\omega(1-\omega^2)(10j-\omega) + \pi\delta(\omega) - 5\pi/101\,(10+j)\delta(\omega+1) - 5\pi/101\,(10-j)\delta(\omega-1)$, determine $F(s)$.
(c) Given $F(s) = 10/s^2(s+1)$, determine $F(j\omega)$.

Solution

(a) Write:

$$F(s) = \frac{A}{s} + \frac{B}{s+10} = \frac{1}{s} - \frac{1}{s+10} = \frac{10}{s(s+10)}$$

therefore from Equation 8-56:

$$F(j\omega) = \frac{10}{j\omega(j\omega+10)} + \pi\delta(\omega)$$

(b) To obtain $F(s)$, zero out the $\delta(\omega)$, $\delta(\omega+1)$, and $\delta(\omega-1)$ terms, then let $\omega = s/j$:

Thus $$F(s) = \frac{10}{s/j(1-(s/j)^2)(10j - s/j)} = \frac{10}{(10s+s^2)(1+s^2)}$$
$$F(s) = \frac{10}{s(s+10)(s^2+1)}$$

(c) Since $F(s)$ has repeated poles on the $j\omega$ axis, we get $f(t)$ first, because Equation 8-56 is not directly applicable in this case. Write

$$F(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} = \frac{10}{s+1} + \frac{10}{s^2} - \frac{10}{s}$$
$$f(t) = 10e^{-t}u(t) + 10tu(t) - 10u(t)$$

We saw that $tu(t) \leftrightarrow j\pi\dot{\delta}(\omega) - 1/\omega^2$ from Example 8-22(e).

Thus $\quad \mathrm{FT}\{f(t)\} = F(j\omega)$

$$= \frac{10}{j\omega + 1} + 10j\pi\dot{\delta}(\omega) - \frac{10}{\omega^2} - 10\pi\delta(\omega) - \frac{10}{j\omega}$$

These considerations cover most of the possible relations between the Fourier transform and the Laplace transform. The fundamental idea here is:

$$\mathrm{LT}\{f(t)\} = \mathrm{FT}\{e^{-\sigma t}f(t)u(t)\} \qquad (8\text{-}57)$$

which is true as long as the Laplace transform of $f(t)$ exists. Working with s terms instead of $j\omega$ terms usually results in simpler algebraic manipulations. In addition, the Laplace transform readily applies to systems with initial conditions, whereas the Fourier transform does not. Although as a rule of thumb working with $F(s)$ is preferred, facility in switching from $F(s)$ to $F(j\omega)$ is essential, especially in the case where frequency response or spectral analysis is at issue. We stress the importance of Equation 8-56. A final example follows.

EXAMPLE 8-29

If $f(t) = e^{+10t}u(t)$, we know that $F(s) = 1/(s - 10)$ is the Laplace transform that exists as long as $\mathrm{Re}(s) = \sigma > 10$. To what might $F(j\omega)$ correspond?

Solution. As we saw previously, if $F(s)$ has RHP poles, $F(j\omega)$ does not exist. Thus the given $f(t)$ does not have a Fourier transform. However, if $g(t) = -e^{10t}u(-t)$, then $G(j\omega) = 1/(j\omega - 10)$ is the Fourier transform of $g(t)$. Note also that $G(j\omega) = F(s)|_{s=j\omega} = F(j\omega)$. So $F(j\omega)$ is not the Fourier transform of the causal time function $f(t)$, but rather, the Fourier transform of the noncausal time function $g(t)$.

Drill Set: Fourier Transforms

1. Use the duality property to determine the Fourier transform of $(\mathrm{Sin}\ t/t)^2$.
2. Use the frequency differentiation property to determine the Fourier transform of $t^3 e^{-5t}u(t)$.
3. Determine the Fourier transform of the periodic signal

$$F(t) = \begin{cases} 1, & 0 < t < 1 \text{ which has a period } T = 3 \\ 2, & 1 < t < 2 \\ 0, & 2 < t < 3 \end{cases}$$

4. Prove that the Fourier transform of the correlation of the $x(t)$ with itself is equal to the magnitude of the Fourier transform squared.
5. If $F(j\omega)$ takes the following forms, determine the corresponding $f(t)$:
   (a) $(j\omega + 1)/(j\omega + 2)(j\omega + 3)$
   (b) $(1 - \omega^2 + j\omega)/(j\omega + 2)(2 - \omega^2 + j\omega)$
   (c) $(j\omega + 5)/(j\omega + 10)(j\omega + 20)^2$
   (d) $j\omega/(j\omega + 1)(j\omega + 2)(j\omega + 3)$

8-6 APPLICATIONS OF FOURIER THEORY

The Fourier transform and its digital counterpart, the discrete Fourier transform, which will be studied in Chapter 9, are widely employed in control systems, communication systems, and signal processing. Many of the signals considered, in particular, in the signal processing area, are in the form of data collected from radar tracks or measurements from various kinds of electromechanical devices. These signals invariably are noisy. Fourier analysis is used with these signals in an attempt to reveal their spectral content. This processing typically employs numerical techniques and the discrete Fourier transform. In addition to processing signals, Fourier analysis is used considerably in the control systems and the communication systems areas. Multiplexing and modulation of communication systems rely heavily on Fourier theory. The design of filters that are employed in control and communication systems is another field in which the Fourier theory is indispensable. Filters can be classified as either analog or digital. Digital filters usually employ the Z transform and analog filters the Fourier transform. As a first example of the application of Fourier theory we briefly consider analog filters.

8-6-1 Filters

One of the most basic applications of the Fourier transform employs the convolution property. Since the Fourier transform of $f(t)*g(t)$ is $F(j\omega)G(j\omega)$, we have an alternative to performing the convolution operation. If $f(t) = h(t)$, the impulse response, then $F(j\omega) = H(j\omega)$, the system function or filter transfer function.

Let $g(t)$ be the system input $x(t)$ and $f*g = h*x = y(t)$, the system output. Then $Y(j\omega) = H(j\omega)X(j\omega)$. To get the output $y(t)$, we need only take the inverse Fourier transform of $Y(j\omega)$. This is another version of the basic input–output relation of the linear systems theory and is useful in comparing real and ideal filters. Consider an ideal low-pass (LP) filter with transfer function $H(j\omega)$ as follows:

$$\begin{aligned} H(j\omega) &= 1, \quad -\omega_0 \le \omega \le \omega_0, \\ &= 0, \quad \text{otherwise} \end{aligned} \qquad (8\text{-}58)$$

$H(j\omega)$ is plotted in Figure 8-20.

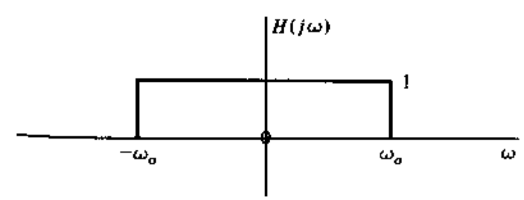


Figure 8-20 Ideal low-pass filter transfer function.

From entry number 9 in Table 8-1, the inverse Fourier transform of this $H(j\omega)$ is $h(t)$

where
$$h(t) = \frac{\text{Sin}\,\omega_0 t}{\pi t} = \frac{\omega_0}{\pi}\,\text{Sinc}\left(\frac{\omega_0}{\pi}t\right) \tag{8-59}$$

This impulse reponse is noncausal. Noncausal systems are not physically realizable; that is, they cannot be built. This is why we say $H(j\omega)$ represents an ideal filter. Although not realizable, we can use $H(j\omega)$ as a standard against which real filters can be compared.

Often the step response instead of the impulse response is used in filter comparisons. We know that $y(t) = x(t)*h(t)$. If the input $x(t)$ is the unit step $u(t)$, then we can call the output $y(t)$, the unit step response $w(t)$. We have:

$$\begin{aligned} w(t) &= u(t)*h(t) \\ &= \int_{-\infty}^{\infty} h(\lambda)u(t-\lambda)\,d\lambda \\ &= \int_{-\infty}^{t} h(\lambda)\,d\lambda \end{aligned} \tag{8-60}$$

and for the ideal low-pass filter

$$w(t) = \int_{-\infty}^{t} \frac{\text{Sin}\,\omega_0\lambda}{\pi\lambda}\,d\lambda \tag{8-61}$$

which is an integral of a Sinc function. In these cases $h(t)$ and $w(t)$ appear as in Figure 8-21. The ripple and overshoot in $w(t)$ on either side of the step discontinuity at $t = 0$ are known as the **Gibbs phenomenon,** named for Josiah Gibbs, a mathematical physicist who studied finite Fourier series approximation theory around 1900.

Now the occurrence for $t < 0$ of the Gibbs phenomenon in $w(t)$ is another indication of the noncausal nature of the ideal low-pass filter. The unit step input is applied at $t = 0$ but the output $w(t)$ has nonzero values for $t < 0$. The noncausal ideal low-pass filter is not physically realizable. To improve the filter somewhat, imagine $w(t)$ to be time shifted to the right by t_0 units. We get a plot of $w(t - t_0)$ which appears in Figure 8-22. This corresponds to a phase shift in $H(j\omega)$ to obtain:

$$\begin{aligned} \overline{H}(j\omega) &= 1e^{-j\omega t_0}, \quad -\omega_0 \le \omega \le \omega_0 \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{8-62}$$

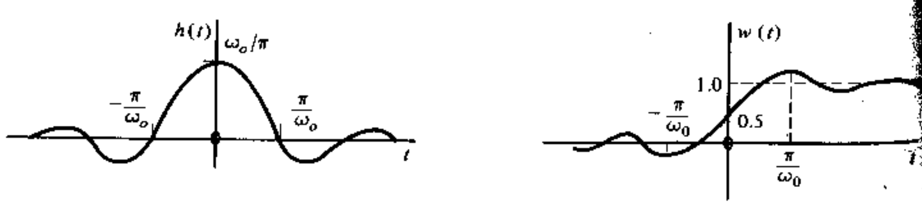


Figure 8-21 Plots of $h(t)$ and $w(t)$ for the ideal LP filter.

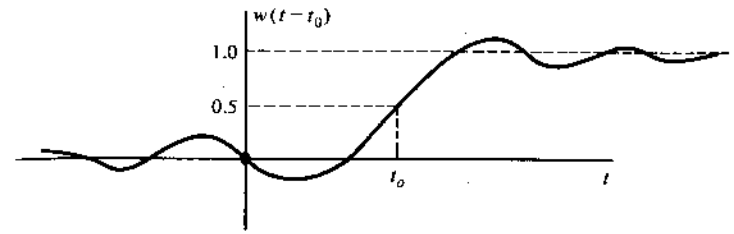


Figure 8-22 Plot of $w(t - t_0)$ for the ideal LP filter.

We can compare these ideal filter characteristics to those of a real low-pass filter, for example, the RC circuit presented in Figure 8-23. We can write:

$$H_{RC}(j\omega) = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j(\omega/\omega_0)} = \frac{1}{\sqrt{1 + \omega^2/\omega_0^2}}\,e^{-j\text{Tan}^{-1}\omega/\omega_0} \tag{8-63}$$

where $\omega_0 = 1/RC$ is the 3 db or half power or cutoff frequency. ω_0 is the bandwidth of the low-pass filter.

This filter has the step response:

$$w(t) = (1 - e^{-\omega_0 t})u(t) \tag{8-64}$$

The step response is plotted in Figure 8-24.

The ideal low-pass (LP) filter has a phase $\overline{\theta}(\omega) = \omega t_0$. From this relation we can define the time delay as follows:

$$t_0 = -\frac{d\overline{\theta}}{d\omega}(\omega)\Big|_{\omega=0} \tag{8-65}$$

Now the phase of the real LP filter is $\theta(\omega) = -\text{Tan}^{-1}\,\omega/\omega_0$. Applying the time delay definition to the real filter, we get:

$$t_0 = -\frac{d\theta}{d\omega}(\omega) = \frac{d}{d\omega}\left(\text{Tan}^{-1}\frac{\omega}{\omega_0}\right) = \frac{1}{1 + (\omega/\omega_0)^2}\frac{1}{\omega_0} \tag{8-66}$$

which when evaluated at $\omega = 0$ becomes:

$$t_0 = \frac{1}{\omega_0} \tag{8-67}$$

Note that the product of the time delay and the filter bandwidth is constant: $t_0\omega_0 = 1$. This reciprocal relationship is important in Fourier transform theory and follows from the time-scaling property of the Fourier transform. If we desire a

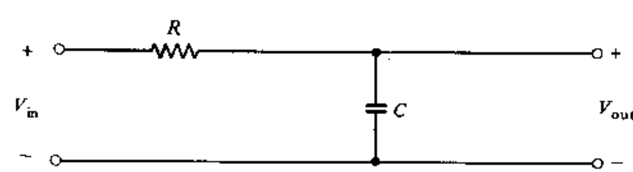


Figure 8-23 RC circuit as low-pass filter.

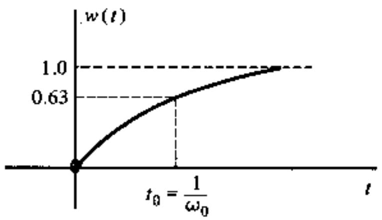


Figure 8-24 Step response of *RC* circuit.

fast response, for instance, a very small value for the t_0 indicated in Figure 8-24, then we must have a very large value for the bandwidth. Consider the rise time, t_r. Although there are a number of different ways to define t_r, we define it as the time from 10 to 90% of the final value.

Thus $$0.1 = 1 - e^{-\omega_0 t_1} \quad \text{and} \quad 0.9 = 1 - e^{-\omega_0 t_2}$$

and $$t_r = t_2 - t_1$$

$$e^{-\omega_0 t_1} = 0.9$$

$$e^{-\omega_0 t_2} = 0.1$$

$$e^{-\omega_0 (t_2 - t_1)} = e^{-\omega_0 t_r} = \tfrac{1}{9}$$

$$\omega_0 t_r = \ln 9 = 2.2$$

Again the reciprocal relationship appears. If we want a very short rise time, for instance, we must have a filter with a very wide bandwidth.

If we consider the real and ideal LP filter representations in the context of the Paley-Wiener criterion, some interesting results follow. Although the proof of this criterion is beyond our scope, the employment of it is fairly straightforward and it can serve as a useful test for causality. The Paley-Wiener criterion can be applied to any Fourier transform to indicate whether or not it corresponds to a causal time function. Let $H(j\omega) = A(\omega) \angle\theta(\omega)$ be the transfer function for a given filter. Then for $H(j\omega) \leftrightarrow h(t)$, if $h(t)$ is causal, the following inequality must hold:

$$\int_{-\infty}^{\infty} \frac{|\ln A(\omega)|}{1 + \omega^2} d\omega < \infty \qquad (8\text{-}68)$$

This criterion is obviously not satisfied by the ideal LP filter since $A(\omega) = 0$ for $|\omega| > \omega_0$ and the natural log of zero is infinity. For the *RC* LP filter, Equation 8.68 is satisfied since $A(\omega) \to 0$ and $\ln A(\omega) \to \infty$ only as $\omega \to \pm\infty$.

The simple *RC* LP filter of Figure 8-23 can be improved by considering more complex circuitry. To improve the filter means to get it closer to the characteristics of the ideal filter. In Figure 8-25 we plot the frequency response magnitude curves for $\bar{H}$ and H_{RC} for positive frequencies. For the ideal LP filter, the spectrum from $0 \le \omega \le \omega_0$ is called the *pass-band* and from $\omega_0 \le \omega < \infty$ is called the *stop-band*. For the real LP filter there is a region around ω_0 called the *transition-band* which can specify the attenuation desired by a certain frequency beyond ω_0. More complex circuitry could result in a LP filter that more nearly approximates the curve for $|\bar{H}|$. These circuits typically employ operational

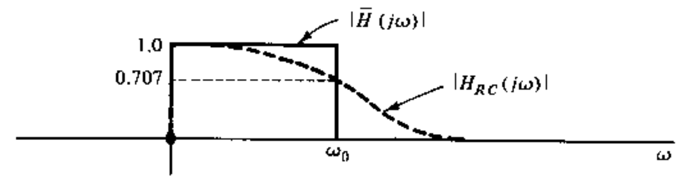


Figure 8-25 Real and ideal LP magnitude frequency response.

amplifiers and various passive circuit elements, usually resistors and capacitors. In practice, inductors are seldom used because of their size and weight.

EXAMPLE 8-30

The circuit shown in Figure 8-26 represents an LP filter that is an improvement over the filter of Figure 8-23. It has a sharper frequency response magnitude curve. Let $R = 1\ \Omega$ and determine C such that the filter has a half power point at 1 kHz.

Solution. At the node labeled V we can write the node equations:

$$\frac{V_{in} - V}{R} = \frac{V}{1/j\omega C} + \frac{V - V_{out}}{R}, \qquad \frac{V - V_{out}}{R} = \frac{V_{out}}{1/j\omega C}$$

then $$V_{in} = (2 + j\omega C)V - V_{out}, \qquad V = V_{out} + j\omega C V_{out}$$

$$= [(2 + j\omega C)(1 + j\omega C) - 1] V_{out}$$

or $$\frac{V_{out}}{V_{in}} = \frac{1}{1 + 3j\omega C + (j\omega C)^2}$$

and $$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{(1 - \omega^2 C^2)^2 + 9\omega^2 C^2}}$$

Setting the magnitude to $1/\sqrt{2}$, we can write:

$$2 = (1 - \omega^2 C^2)^2 + 9\omega^2 C^2 = 1 + 7\omega^2 C^2 + \omega^4 C^4$$

Let $x = \omega^2 C^2$ and write:

$$x^2 + 7x - 1 = 0 \quad \text{or} \quad x = 0.14,$$

taking only the positive value. Now $\omega = 2\pi f = 2\pi 10^3$.

Thus $$C^2 = \frac{0.14}{4\pi^2 10^6} \quad \text{or} \quad C = 59.5 \mu F$$

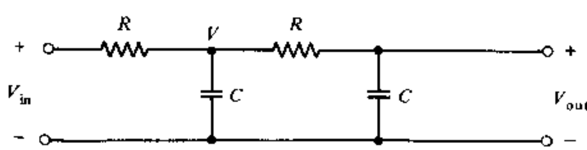


Figure 8-26 Low-pass filter for Example 8-30.

In the practical design of analog filters today engineers most often use the Butterworth, Bessel, or Chebyshev filters. Each of these designs is an approximation to the ideal LP filter. Each is "optimal," or best, but in a different sense. A detailed consideration of these filters is beyond our present scope. We mention only a few general characteristics:

1. The Butterworth filter is noted for having a response in the pass-band that is optimum in that it is as flat as possible for a given filter order. (**Filter order** is just the degree of the polynomial in the filter transfer function. The order is generally the number of energy storage devices required.)
2. The Bessel filter has a phase response that is as linear as possible for a given filter order.
3. The Chebyshev filter maintains a specified amplitude response in a given range of the pass-band. Although it has ripples in the pass-band, it is monotonic in the stop-band and yields maximum attenuation for a specified filter order. The Chebyshev and Bessel filter are so named because of their characterization in terms of Chebyshev polynomials and Bessel functions.

So far, the discussion on filters has centered around the LP filter. By employing some simple transformations, however, we can convert a given low-pass filter into a high-pass, band-pass, or notch filter. Assume we have a normalized LP filter described by $H(j\omega)$, where $\omega = 1$ is the cutoff frequency and $|H(j0)| = 1$. We can normalize the frequency by replacing ω by ω/ω_0. Replace ω by $-1/\omega$ and we get a high-pass (HP) filter. Replace ω by $(\omega^2 - \omega_1\omega_2)/\omega(\omega_2 - \omega_1)$ where ω_1 and ω_2 are the lower and upper cutoff frequencies and we get a band-pass filter with bandwidth $= \omega_2 - \omega_1$. Finally, replace ω by $\omega(\omega_2 - \omega_1)/(\omega_1\omega_2 - \omega^2)$ and we obtain a notch filter, where ω_1 and ω_2 are the cutoff frequencies. The network synthesis problem of determining the proper *RLC* component values or proper op-amp configuration for a given transfer function is a more difficult task and will not be pursued here. We merely note in concluding this discussion on filters that the Fourier transform does play an important role in both the analysis and the synthesis of filters and that filters of a variety of types are being used more and more in technical devices of all kinds.

8-6-2 Amplitude Modulation

Consider an application of Fourier analysis to amplitude modulation (AM). Although there are many ways to indicate the AM modulated signal, we will employ the form:

$$f(t) = (1 + ms(t)) \operatorname{Cos} \omega_c t \qquad (8\text{-}69)$$

This represents what is called double-sideband amplitude modulation. Assume the audio signal is $s(t)$. Also, assume it is bounded in magnitude by 1.0 and its highest frequency is $\omega_s \ll \omega_c$, where ω_c is called the carrier frequency. Let m, the

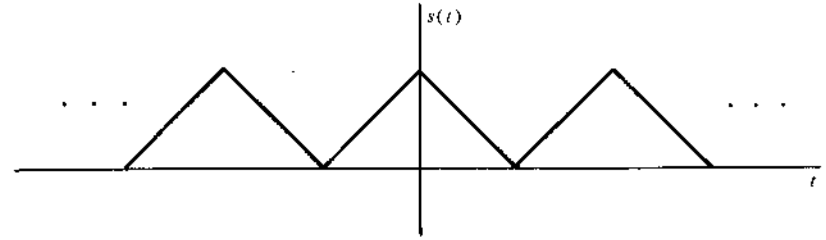


Figure 8-27 Triangular pulse train.

index of modulation, range from 0 to 1. Now the term $1 + ms(t)$ varies slowly compared to $\operatorname{Cos} \omega_c t$, which means that we can view $1 + ms(t)$ as an envelope to $\operatorname{Cos} \omega_c t$. The term $1 + ms(t)$ functions as the amplitude of the $\operatorname{Cos} \omega_c t$ sinusoid. As $s(t)$ varies, this sinusoid then has an amplitude that varies with time: The amplitude is said to be modulated by the variations in $s(t)$. For instance, if $s(t)$ is a triangular pulse train as in Figure 8-27, then $f(t)$ might appear as in Figure 8-28. To consider the spectrum of $f(t)$ in an explicit fashion, let $s(t) = \operatorname{Cos} \omega_s t$, a very simple audio signal, but sufficient for purposes of illustration:

$$f(t) = (1 + m \operatorname{Cos} \omega_s t) \operatorname{Cos} \omega_c t$$

$$= \operatorname{Cos} \omega_c t + \frac{m}{2} \operatorname{Cos} (\omega_c + \omega_s)t + \frac{m}{2} \operatorname{Cos} (\omega_c - \omega_s)t \qquad (8\text{-}70)$$

Expressing these cosines as complex exponentials, we can write the following:

$$f(t) = \tfrac{1}{2}e^{j\omega_c t} + \tfrac{1}{2}e^{-j\omega_c t} + \frac{m}{4}e^{j(\omega_c+\omega_s)t} + \frac{m}{4}e^{-j(\omega_c+\omega_s)t} + \frac{m}{4}e^{j(\omega_c-\omega_s)t} + \frac{m}{4}e^{-j(\omega_c-\omega_s)t} \qquad (8\text{-}71)$$

This time function has the complex Fourier line spectrum indicated by Figure 8-29. Note that all these frequencies are relatively "high" frequencies, which are essential for long-distance transmission. After this signal is transmitted it is necessary to recover $s(t)$, the signal of interest. This is called **demodulation.** There are many schemes available to do this, the simplest of which probably is

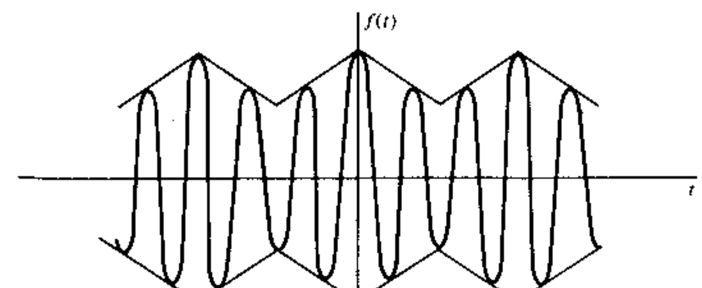


Figure 8-28 Plot of a typical A.M. signal.

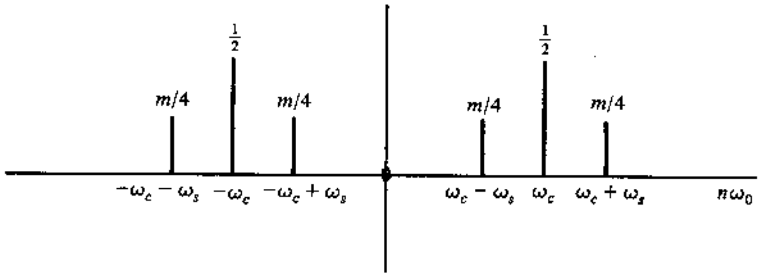


Figure 8-29 Spectrum of an A.M. signal.

the detector or envelope demodulator, a circuit that is shown in Figure 8-30. Referring to $f(t)$ in Figure 8-28, we find that if this signal is input to the circuit of Figure 8-30, the rectifier will pass the positive portion of $f(t)$ and the RC filter with $RC = 1/\omega_s$, will transmit only the envelope of this positive portion. The output will then be a reasonable facsimile of $s(t)$.

Now let $s(t)$ be generalized somewhat to be a time signal that is band-limited—instead of just a single sinusoid. Such an $s(t)$ will have a Fourier transform $S(j\omega)$ perhaps like the one in Figure 8-31. Band-limitedness in reality is a fiction because it implies an $s(t)$ that is of infinite duration. Often, however, a signal will display a spectrum that is negligibly small outside a certain band. These signals can be approximated by a band-limited spectrum, which is useful at least for the purpose of illustration. Consider the Fourier transform of the $f(t)$ of Equation 8-69.

$$F(j\omega) = \pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c) + \frac{m}{2}S(j(\omega - \omega_c) + \frac{m}{2}S(j(\omega + \omega_c) \qquad (8\text{-}72)$$

This equation indicates that $S(j\omega)$ is shifted to yield a spectrum like the one in Figure 8-32. Again, once the modulated signal is transmitted, there is the need to demodulate. Again, a circuit similar to the circuit of Figure 8-30 will process the received $f(t)$ signal and yield a good approximation to $s(t)$ at the circuit output.

So far, all this AM discussion has focused on what is called **asynchronous** amplitude modulation. This refers primarily to the demodulation side of the system. In asynchronous demodulation there is no need to have available an oscillator synchronized to the carrier frequency ω_c. We merely send $f(t)$ through a circuit like that in Figure 8-30. Why is this advantageous? Employing an oscillator at the receiver synchronized to ω_c is a costly venture as well as a tricky

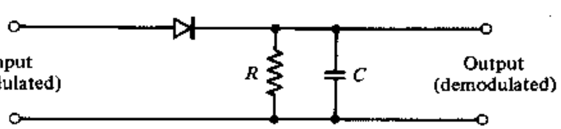


Figure 8-30 Detector demodulator.

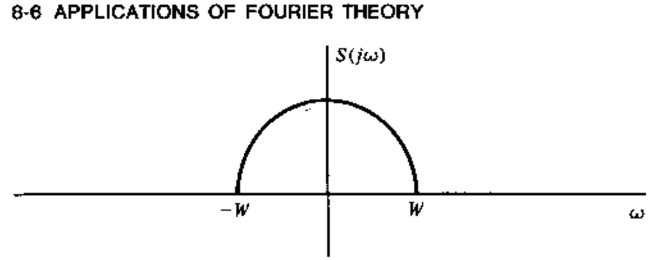


Figure 8-31 Transform of a band-limited signal.

stabilization problem. For accurate signal recovery, the receiver oscillator must be set and maintained at the exact frequency and phase of the oscillator at the transmitter. This is called **synchronous** amplitude modulation and although it is practically more difficult to implement than an asynchronous system, it is conceptually appealing because of its simplicity. We will present its fundamentals which will then be useful in a discussion of ***multiplexing***.

In synchronous AM the transmitted signal is typically expressed as:

$$f(t) = s(t)\ \mathrm{Cos}\ \omega_c t \qquad (8\text{-}73)$$

Again, assume $\omega_s \ll \omega_c$, where ω_s is the highest frequency in $s(t)$. We illustrate this with the band-limited signal whose spectrum is represented in Figure 8-31.

Then
$$F(j\omega) = \frac{S(j[\omega + \omega_c]) + S(j[\omega - \omega_c])}{2} \qquad (8\text{-}74)$$

A plot of this would be similar to the one in Figure 8-32 except for the impulses at $\omega = \pm\omega_c$. Transmission of $f(t)$ proceeds just like in the asynchronous case. The major difference occurs at the demodulation receiver side of the operation. To demodulate the received $f(t)$, we multiply by Cos $\omega_c t$ and then perform a LP filtering operation. The frequency of this cosine must be exactly the same as (synchronized to) the frequency of the cosine on the transmitting side of the operation. This synchronization problem, keep in mind, is the difficult part. Here is the way it works:

$$f(t)\ \mathrm{Cos}\ \omega_c t = s(t)\ \mathrm{Cos}^2\ \omega_c t = s(t)\left\{\frac{1 + \mathrm{Cos}\ 2\omega_c t}{2}\right\} \qquad (8\text{-}75)$$

and this has the Fourier transform:

$$\mathrm{FT}\{f(t)\ \mathrm{Cos}\ \omega_c t\} = \frac{S(j\omega)}{2} + \frac{S(j[\omega + 2\omega_c]) + S(j[\omega - 2\omega_c])}{4} \qquad (8\text{-}76)$$

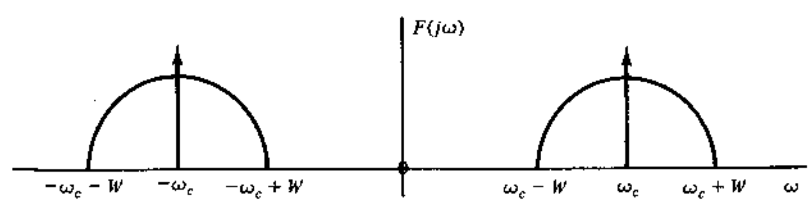


Figure 8-32 Transform of a modulated signal.

This spectrum appears as in Figure 8-33. Note that the shape of the part of the spectrum centered about the origin is exactly the same as the shape of the spectrum in Figure 8-31. This implies that we need only perform a LP filtering operation to recover the original $s(t)$. An ideal LP filter with cutoff frequency $\geq W$ will suffice. Deviations from ideality of course will produce some distortion in the received signal. This is where trade-offs become essential.

8-6-3 Multiplexing

Multiplexing is a technique of simultaneous transmission of a number of different signals over a single channel.

We consider here a type of multiplexing called frequency-division-multiplexing (FDM). In this process each input spectrum is assigned a distinct frequency band. The modulation and demodulation in an FDM system are both based on frequency translation. This is very similar to the synchronous AM discussed previously. Although large numbers of signals can be handled simultaneously, we examine only two of them. This will illustrate the procedure without too much clutter.

Imagine that we have $s_1(t)$ and $s_2(t)$ as information signals. Assume each is band-limited with Fourier transforms like those in Figure 8-34. If we modulate $s_2(t)$ with Cos $\omega_2 t$ and $s_1(t)$ with Cos $\omega_1 t$, we can form the signal $x(t)$:

$$x(t) = s_1(t) \text{ Cos } \omega_1 t + s_2(t) \text{ Cos } \omega_2 t \tag{8-77}$$

then

$$X(j\omega) = \frac{S_1(j[\omega + \omega_1]) + S_1(j[\omega - \omega_1])}{2} + \frac{S_2(j[\omega + \omega_2]) + S_2(j[\omega - \omega_2])}{2} \tag{8-78}$$

which has a spectrum similar to that in Figure 8-35. It is important here to make sure that the spectra do not overlap; that is, $\omega_1 + W_1 < \omega_2 - W_2$. The signal $x(t)$ is then transmitted and received and must be demodulated to recover $s_1(t)$ and $s_2(t)$. In view of $X(j\omega)$ in Figure 8-35 to capture the spectrum of $s_1(t)$, we would need a band-pass filter centered around ω_1 and for $s_2(t)$ a band-pass filter centered around ω_2. Then, with the output of each of these band-pass filters, we would proceed as in the case of synchronous AM demodulation: Multiply by Cos $\omega_1 t$ (or Cos $\omega_2 t$) and pass through LP filters. The overall system of

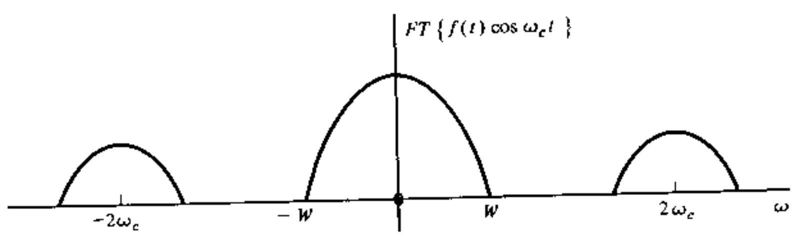


Figure 8-33 Spectrum of the synchronous demodulator.

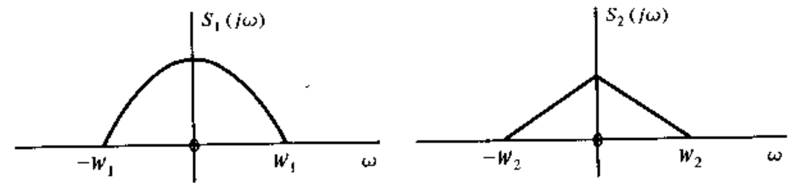


Figure 8-34 Transform of S_1 and S_2 for the FDM system.

modulation, multiplexing, transmission, demultiplexing, and demodulation is illustrated in Figure 8-36.

Of course, the FDM system presented here is idealistic. As in the synchronous AM system, we need ideal LP filters, one with a cutoff frequency $\geq W_1$, the other with the cutoff $\geq W_2$. The ω_1 and ω_2 frequencies on the demodulation side must be perfect matches of those on the modulation side. The band-pass filters must also be ideal. The band-limitedness assumption is very troublesome in FDM systems. Since no real signals are ever band-limited, the $X(j\omega)$ spectrum will display overlap between the distinct frequency bands. This overlap results in a phenomenon called *aliasing*, which we will consider in greater detail in Chapter 9. Although many simplifying assumptions are involved in this discussion, these are the rudiments of frequency-division-multiplexing. Extension from two to n signals follows in a straightforward manner.

8-6-4 Frequency Modulation

Fourier methods can also be usefully applied to other forms of modulation. *Angle modulation* consists of two basic types: phase modulation (PM) and frequency modulation (FM). The preceding discussion on AM started with the typical carrier signal $f(t) = A$ Cos $\omega_c t$, where $A = 1 + ms(t)$ in the asynchronous case and $A = s(t)$ in the synchronous case represented the time varying sinusondal amplitude. Now let:

$$f(t) = A \text{ Cos } \theta(t) = A \text{ Cos } (\omega_c t + \theta_c) \tag{8-79}$$

where θ_c is the phase and ω_c is the frequency of the carrier. Assume A is constant. In PM the phase is modulated so that:

$$\theta_c = \theta_c(t) = \theta_0 + k_1 \mu_1(t) \tag{8-80}$$

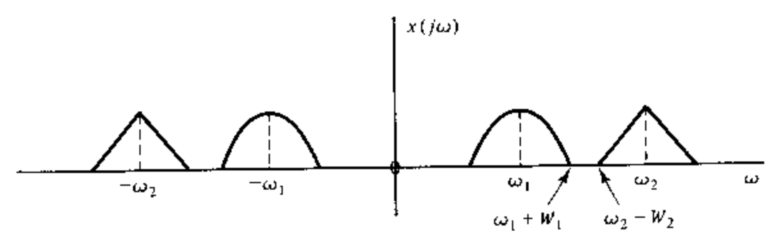


Figure 8-35 Spectrum of $x(t)$ for the FDM system.

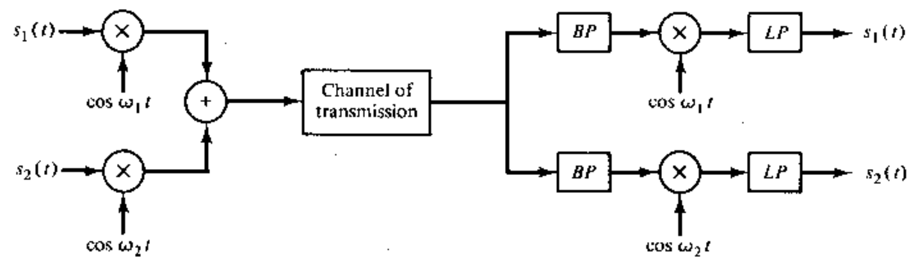


Figure 8-36 Frequency-division-multiplexing (FDM) system.

In FM the frequency is modulated. This is done by letting:

$$\theta(t) = \omega_c t + k_2 \int_0^t \mu_2(t)\, dt \tag{8-81}$$

Then the instantaneous frequency $\omega_i(t)$ is:

$$\omega_i(t) = \frac{d\theta(t)}{dt} = \omega_c + k_2\mu_2(t) \tag{8-82}$$

Since the mathematics of general angle modulation can get very complex, consider here only some basic FM ideas. Also, since even basic FM analysis is difficult, we deal with the special case where:

$$\mu_2(t) = \text{Cos}\ \omega_0 t \tag{8-83}$$

In spite of the simplicity of this assumption, it should give some good insight into the concept of frequency modulation. From Equation 8-81 we obtain:

$$\theta(t) = \omega_c t + \frac{k_2}{\omega_0} \text{Sin}\ \omega_0 t = \omega_c t + k\ \text{Sin}\ \omega_0 t \tag{8-84}$$

Then

$$\begin{aligned} f(t) &= A\ \text{Cos}\ (\omega_c t + k\ \text{Sin}\ \omega_0 t) \\ &= A\ \text{Re}\ \{e^{j(\omega_c t + k\text{Sin}\omega_0 t)}\} \\ &= A\ \text{Re}\ \{e^{j\omega_c k} e^{jk\text{Sin}\omega_0 t}\} \end{aligned} \tag{8-85}$$

The second complex exponential can be expressed in terms of the Bessel functions:

$$e^{jk\text{Sin}\omega_0 t} = \sum_{n=-\infty}^{\infty} J_n(k) e^{jn\omega_0 t} \tag{8-86}$$

Bessel functions, $J_n(k)$, arise as solutions of certain kinds of differential equations. They are tabulated functions, available in any extensive tables of mathematical formulas. The Fourier transform of $f(t)$ can now be written:

$$F(j\omega) = A\ \text{Re}\left\{\sum_{n=-\infty}^{\infty} J_n(k)\delta(\omega - \omega_c - n\omega_0)\right\} 2\pi \tag{8-87}$$

and since the right-hand side is purely real, we can write:

$$F(j\omega) = A \sum_{n=-\infty}^{\infty} J_n(k)\delta(\omega - \omega_c - n\omega_0) \cdot 2\pi \tag{8-88}$$

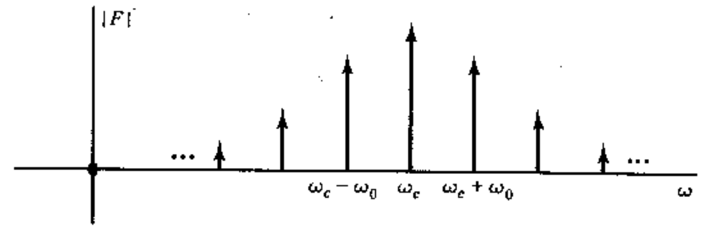


Figure 8-37 Spectrum for an FM signal.

Then the FM spectrum consists of an infinite number of impulses centered around $\omega = \omega_c$ and would appear as in Figure 8-37. Each impulse is weighted with a value obtained from Bessel functions. Such signals are called wide-band FM signals since there are now an infinite number of sidebands. The weights of the impulses representing the sidebands, however, become negligibly small after a very short excursion on either side of ω_c.

These are just the very basic ideas associated with frequency modulation. We have generated a signal that is of sufficiently high frequency to be transmitted. Then, of course, it will need to be received and demodulated. FM demodulation is a more difficult task than AM demodulation. We will leave it for a more involved study in the area of communication systems.

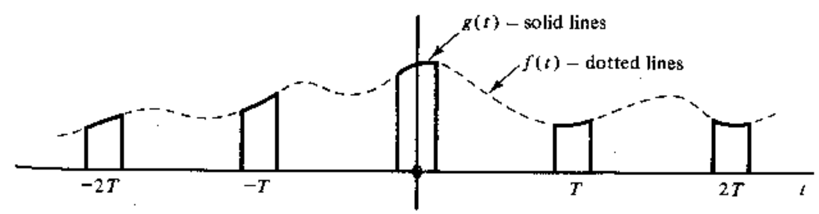
8-6-5 The Sampling Theorem

As a final application of Fourier theory we consider a theorem that has had an important impact on very large sectors of the technological world, especially the digital areas. The theorem discussed in this continuous time Fourier chapter can provide the introduction into the next chapter which deals with discrete time Fourier analysis. Very often discrete signals are obtained from continuous signals via a process of sampling.

There is a famous theorem called the **Shannon sampling theorem** which delimits the sampling process. It says that if $f(t) \leftrightarrow F(j\omega)$ is band-limited by $-\omega_B < \omega < \omega_B$, then $f(t)$ is recoverable from its samples $f(nT)$, $n = 0, \pm 1, \ldots$ if:

$$\omega_0 > 2\omega_B \tag{8-89}$$

where $\omega_0 = 2\pi/T$ is the sampling frequency. To illustrate what is involved here, we can employ Fourier transform ideas. Let $f(t)s(t) = g(t)$, the sampled function. Assume that $f(t)$ is continuous and that $s(t)$ is a sampling function



consisting of a train of narrow pulses of amplitude 1, separated by T time units. Then $g(t)$ might appear as in Figure 8-38. Since $f(t)$ is band-limited, in order to illustrate the sampling theorem, we assume $F(j\omega)$ is as in Figure 8-39. Now since $s(t)$ is periodic, we know that it has a complex exponential Fourier series representation:

$$s(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \tag{8-90}$$

and

$$S(j\omega) = \sum_{n=-\infty}^{\infty} 2\pi c_n \delta(\omega - n\omega_0) \tag{8-91}$$

From the frequency domain convolution property we know that:

$$g(t) = f(t)s(t) \leftrightarrow G(j\omega) = \frac{F(j\omega) * S(j\omega)}{2\pi}$$

Therefore

$$G(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\beta)S(j(\omega - \beta))d\beta \tag{8-92}$$

Since S from Equation 8-91 contains impulses, Equation 8-92 is easily integratable. We obtain:

$$G(j\omega) = \sum_{n=-\infty}^{\infty} c_n F(j(\omega - n\omega_0)) \tag{8-93}$$

illustrated in Figure 8-40. To recover the exact shape of $F(j\omega)$ we only need to pass $g(t)$ through an ideal LP filter whose cutoff frequency is ω_B. Note, however, that we are assuming $\omega_0 > 2\omega_B$ exactly as the sampling theorem requires. If $\omega_0 < 2\omega_B$, we get overlap in the spectrum of $G(j\omega)$. Then $f(t)$ is not recoverable. This critical frequency, ω_0, which is the sampling frequency, is also sometimes called the **Nyquist frequency.**

To conclude this section we mention an application of the sampling theorem to communication systems. Modern data communication and telephone systems often employ a technique called **time-division-multiplexing (TDM).** In this process a large number of samples of different signals are transmitted over the same channel. Upon reception, complex synchronization equipment is needed to separate the various signals. Then from the samples the original signals can be recovered. This requires, however, a Nyquist frequency not just of ω_0, but if we multiplex, say, N signals, we will need a Nyquist frequency of $N\omega_0$. Further aspects of sampling will be considered in the next chapter when we deal with the discrete Fourier transform.

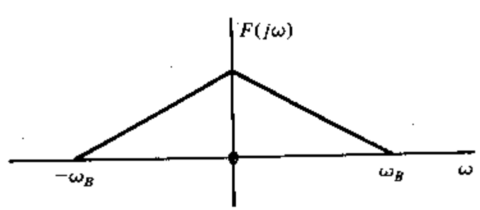


Figure 8-39 Spectrum of $f(t)$.

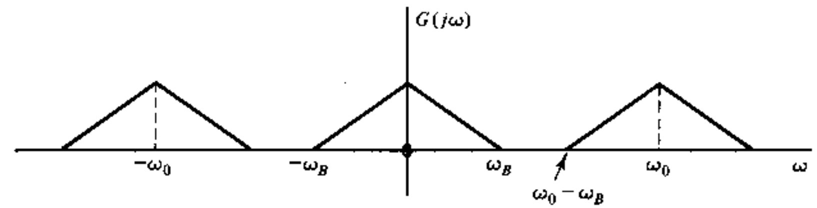


Figure 8-40 Spectrum of $g(t)$, the sampler output.

SUMMARY

This chapter considered the Fourier analysis of continuous time functions $f(t)$. The spectral content indicating the frequencies of significance contained in $f(t)$ played a large role in this chapter. For periodic $f(t)$'s the spectral content is revealed via the Fourier series analysis. We discussed the trigonometric, generalized, and exponential Fourier series. From the exponential Fourier series we developed the Fourier transform to deal with nonperiodic signals.

In the Fourier transform development a number of illustrative examples were worked out, then many properties were considered. Most of these properties were based on the defining integrals:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega$$

Following the properties of the Fourier transform, a brief section was presented in which the Fourier and Laplace transforms were compared.

A few applications of the Fourier transform were considered in the last section. Among these applications were some brief discussions of analog filter design and amplitude and frequency modulation. Also, we discussed some multiplexing ideas: frequency- and time-division-multiplexing. Then we concluded with an introduction to the sampling theorem. Further applications of the Fourier transform will be dealt with in the next chapter when we study the Fourier analysis of discrete time signals.

PROBLEMS

8-1. Determine the trigonometric Fourier series expansion of the periodic signal $f(t)$ where $f(t) = t^2$ for $0 < t < 2\pi$ and $T = 2\pi$.

8-2. Determine the exponential Fourier series expansion of the periodic signal $f(t)$ where $f(t) = e^{-t}$ for $0 < t < 2$ and $T = 2$.

8-3. Consider the following set of basis functions, orthonormal over $0 \leq t < \infty$.

$$\phi_1(t) = \sqrt{2}e^{-t}, \quad t \geq 0$$
$$= 0, \quad t < 0$$

and $$\phi_2(t) = 6e^{-2t} - 4e^{-t}, \quad t \ge 0$$
$$= 0, \quad t < 0$$

Let
$$f(t) = 2e^{-3t}, \quad t \ge 0$$
$$= 0, \quad t < 0$$

Determine $\hat{f}(t) = c_1\phi_1(t) + c_2\phi_2(t)$ and examine the accuracy of this approxima- tion using Parseval's relation.

8-4. A certain system has a system function:

$$H(j\omega) = \frac{1}{j\omega + 10}$$

Assume that the input $x(t)$ is periodic with period 2

and $$x(t) = \begin{cases} 10, & 0 < t < 1 \\ 5, & 1 < t < 2 \end{cases}$$

Determine $y(t)$, the system output.

8-5. Consider

$$f(t) = f(t + 2)$$
$$= 1 - |t|, \quad \text{for } -1 < t < 1$$

(a) Sketch $f(t)$, $f'(t)$, $f''(t)$. Do not forget the singularity functions that ma- occur.
(b) Let

$$f''(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

Determine the Fourier series coefficients C_n.
(c) Find the relation between the Fourier series coefficients for $f(t)$, $f'(t)$, an- $f''(t)$.
(d) Express $f(t)$ in an exponential and trigonometric Fourier series.
(e) Find the Fourier transform of one period of $f(t)$, $f'(t)$, and $f''(t)$.
(f) Evaluate the exponential Fourier coefficients from the transform of a singl- period of $f(t)$, $f'(t)$, and $f''(t)$. Compare these with the results of parts (b) an- (c).

8-6. Given the transform pair:

$$e^{-\pi t^2} \leftrightarrow e^{-\omega^2/4\pi}$$

Evaluate:

(a) $\int_0^{\infty} e^{-t^2}\, dt$

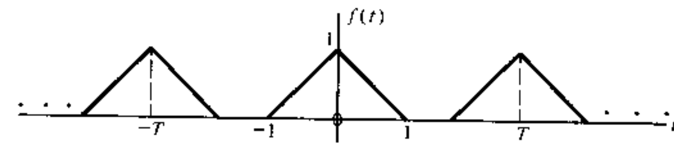
(b) $\int_0^{\infty} t^2 e^{-t^2}\, dt$

8-7. Find the Fourier transform and plot $|F(j\omega)|$ versus ω for the following
(a) $f(t) = e^{-2t}u(t) - e^{2t}u(-t)$

(b) $f(t) = \dfrac{\text{Sin } 2\pi t}{2\pi t}(4 + \text{Cos } 20t)$

(c) $f(t) = \dfrac{\text{Sin } \pi t}{\pi t}\text{Cos}^2\, 50t$

8-8. Consider the following periodic $f(t)$:



For this $f(t)$,

$$c_n = \frac{\text{Sin}^2(n\pi/T)}{n^2\pi^2/T}$$

(a) Assume T is very large so that $\text{Sin}(n\pi/T) \approx n\pi/T$. Determine T such that $c_5 = 1/100$.
(b) Now let $T = 3$ and determine the energy in $f(t)$ and the energy in

$$\hat{f}(t) = \sum_{n=-2}^{2} c_n e^{jn\omega_0 t}$$

Compare results.

8-9. Consider a periodic

$$f(t) = A, \quad 0 < t < \frac{T}{2}$$
$$= 4, \quad \frac{T}{2} < t < T$$

T is the period.
(a) Determine A if $c_1 = 2/\pi\,\underline{|90°}$
(b) Determine A if $c_1 = 2/\pi\,\underline{|-90°}$
(c) Determine c_2 if $A = 10$.

8-10. A certain periodic $f(t)$ has the Fourier series coefficients: $c_0 = 5$; $c_1 = 2 + 2j$; $c_{-1} = 2 + 2j$; all other c's $= 0$. We can write

$$f(t) = f_1(t) + jf_2(t),$$

a complex time function. Determine $f_1(t)$ and $f_2(t)$.

8-11. Let

$$f(t) \leftrightarrow F(j\omega) = R(\omega) + jX(\omega)$$

Assume $f(t) = 0$, for $t < 0$. Let:

$$X(\omega) = e^{-\omega}$$

and $$R(\omega) = \int_0^{\infty} g(t)\,\text{Cos}\,\omega t\, dt$$

Determine $g(t)$. [This problem is nontrivial. *Hint:* Solve for $R(\omega)$ as a function of $X(\omega)$.]

8-12. Determine $G(j\omega)$, where $g(t) = d^2/dt^2\,(f(t)\,\text{Cos}\,2t)$ and $g(t) \leftrightarrow G(j\omega)$:

$$f(t) \leftrightarrow F(j\omega) = \frac{5}{5 + j\omega}$$

8-13. Let $f(t)$ be two impulses:

$$f(t) = A\delta(t - t_1) + B\delta(t - t_2)$$

Assume:

$$F(j\omega) = e^{-j\omega}\,\text{Cos}\,\omega$$

Determine suitable values for A, B, t_1, and t_2.

8-14. Let:

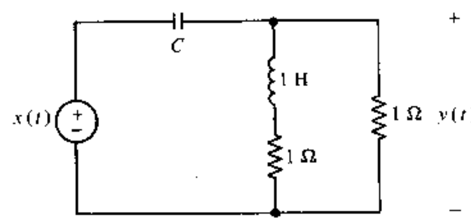
$$f(t) = u(t) - u(t - 1) + u(t - 2) - u(t - 3)$$

(a) Determine $F(j\omega)$.
(b) Let:

$$g(t) = f(t)$$

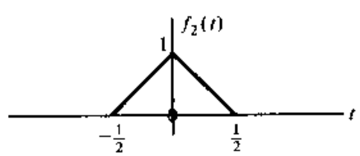
Determine $G(j\omega)$.

8-15. Consider the following *RLC* circuit.



(a) Determine $H(j\omega) = Y(j\omega)/X(j\omega)$ as a function of C.
(b) Determine C such that $|H(j\omega)| = 1$ at $\omega = \sqrt{2}\,r/s$.
(c) Determine the phase of $H(j\omega)$ at $\omega = \sqrt{2}\,r/s$ and $C = 1F$.

8-16. Let $f_1(t) = \text{Cos}\,t$ and let $f_2(t)$ be as follows:



Determine the Fourier transform of:

$$f(t) = f_1(t)f_2(t)$$

8-17. Let:

$$F(j\omega) = \frac{1 + j\omega}{8 + 6j\omega + (j\omega)^2} \leftrightarrow f(t)$$

and

$$\ddot{g}(t) + 2\dot{g}(t) + g(t) = f(t)$$

Determine $G(j\omega)$.

8-18. A certain system:

$$H(j\omega) = \frac{100}{(j\omega)^2 + 0.2j\omega + 100}$$

has an input:

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\,\text{Sin}\,\frac{10}{3}t + \frac{2}{3\pi}\,\text{Sin}\,10t + \frac{2}{5\pi}\,\text{Sin}\,\frac{50}{3}t + \cdots$$

Assume that the output is

$$y(t) \approx k\sin(\omega_0 t + \theta)$$

Determine k, ω_0, and θ.

8-19. Let:

$$f(t) = 3u(t) - 3u(t - 2) + 3u(t - 3) - 3u(t - 4)$$

Determine the energy spectral density for $f(t)$

and

$$\int_{-\infty}^{\infty} F(j\omega)F(-j\omega)\,d\omega$$

8-20. For a certain $f(t)$:

$$F(j\omega) = \frac{1}{2 + j\omega}$$

(a) Determine $\int_{-\infty}^{\infty} f(t)\,dt$.

(b) Let:

$$F_1(j\omega) \leftrightarrow f_1(t) = \int_{-\infty}^{t} f(\lambda)\,d\lambda$$

Determine $F_1(j\omega)$.

8-21. Let:

$$f(t) = u(t + 1) - u(t - 1)$$

(a) Show that $f(t)$ can be expressed as follows:

$$f(t) = k\int_0^{\infty} \frac{\text{Sin}\,\omega}{\omega}\,\text{Cos}\,\omega t\,d\omega$$

(b) Determine k.

8-22. Let:

$$f(t) = \frac{\text{Sin}^2\,t}{t^2}$$

Using Parseval's equation, determine $\int_{-\infty}^{\infty} f(t)^2\,dt$.

8-23. Determine the autocorrelation function $R(\tau)$ for:

$$f(t) = \text{Au}\left(t + \frac{a}{2}\right) - \text{Au}\left(t - \frac{a}{2}\right)$$

$$R(\tau) \triangleq \int_{-\infty}^{\infty} f(\lambda - \tau) f(\lambda)\, d\lambda$$

(This is just the correlation of a function with itself.)

8-24. Show that the inverse Fourier transform of $E(\omega) = F(j\omega)F(-j\omega)$ is the autocorrelation function of $f(t)$. Show that the autocorrelation function is an even function of its argument.

8-25. Let:

$$F(j\omega) = \frac{10 + 2j\omega}{(5 + 3j\omega)(2 + j\omega)}$$

Determine the Fourier transform of:

(a) $f(t) \operatorname{Sin} \omega_0 t$ **(b)** $f(t/5)$

(c) $t^3 f(t)$ **(d)** $d^3/dt^3 f(t)$

8-26. Assume $f(t) \leftrightarrow F(j\omega)$. Show that:

$$|F(j\omega)| \leq \frac{1}{|\omega|^n} \int_{-\infty}^{\infty} \left| \frac{d^n f}{dt^n} \right| dt, \qquad \text{for } n = 0, 1, 2, \cdots$$

where $|\omega|^0 = 1$ and $|d^0 f/dt^0| = |f|$.

8-27.

$$F_1(j\omega) \leftrightarrow f_1(t)$$

and

$$F_2(j\omega) \leftrightarrow f_2(t)$$

Let:

$$F_1(j\omega) = \frac{1}{(5 + j\omega)(2 + j\omega)}$$

Let:

$$F_2(j\omega) = F_1(j(\omega + \omega_0)) + F_1(j(\omega - \omega_0))$$

Determine $f_2(t)$.

8-28. Determine and sketch the autocorrelation function for $f(t)$

where

$$\begin{aligned} f(t) &= 0, \quad t < -\frac{1}{2} \\ &= -10, \quad -\frac{1}{2} < t < 0 \\ &= 10, \quad 0 < t < \frac{1}{2} \\ &= 0, \quad t > \frac{1}{2} \end{aligned}$$

Determine the autocorrelation function of $\bar{f}(t)$. $\bar{f}(t)$ is the periodic version of $f(t)$. Assume $T = 2$.

8-29. The power spectral density of a continuous random process $x(t)$ is defined as:

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau}\, d\tau$$

and the cross-spectral densities for processes $x(t)$ and $y(t)$ as:

$$S'_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j\omega\tau}\, d\tau$$

and

$$S_{yx}(\omega) = \int_{-\infty}^{\infty} R_{yx}(\tau) e^{-j\omega\tau}\, d\tau$$

Using the properties of correlation functions, show:

(a) $S_{xx}(\omega) = S_{xx}(-\omega)$

(b) $P_{av} = x^2(t) = \int_{-\infty}^{\infty} S_{xx}(\omega)\, df$

(c) $S_{xy}(\omega) = S_{yx}^*(\omega)$

(d) $S_{xx}(\omega_0)\ \Delta f$ is the total average power from a narrow bandpass filter of bandwidth Δf about f_0. Demonstrate that all these properties hold for a process with autocorrelation function $R_{xx}(\tau) = e^{-2|\tau|}$ and cross-correlation function $R_{xy}(\tau) = 3e^{-|\tau-4|}$.

8-30. Given the input to a linear system with system function $H(j\omega)$ is zero-mean noise with a power spectral density $S_{xx}(\omega)$, transform the time domain results to show:

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$$

$$S_{xy}(\omega) = S_{xx}(\omega) H(\omega)$$

and

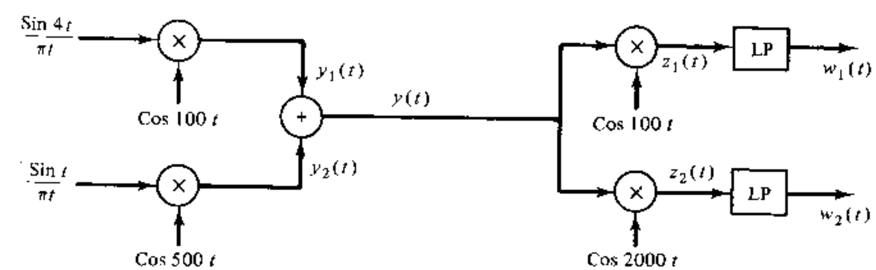
$$S_{yx}(\omega) = S_{xx}(\omega) H(-\omega)$$

8-31. Given a linear system with system function $H(\omega) = 1/2 + j\omega$ has as its input white noise with actual mean square fluctuations $\tilde{x}^2(t) = 20$ and power spectral density $S_{xx}(\omega) = 2$, find

(a) $S_{yy}(\omega)$, $S_{xy}(\omega)$, and $y^2(t) = \int_{-\infty}^{\infty} S_{yy}(\omega)\, df$

(b) Do the spectral quantities have the desired properties?

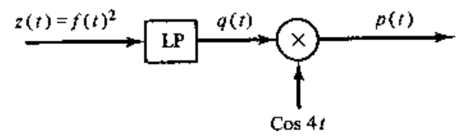
8-32. Consider the frequency-division-multiplixing system:



The blocks labeled LP are ideal low-pass filters. The top filter has a bandwidth $-4 \leq \omega \leq 4$ and the bottom filter has a bandwidth $-1 \leq \omega \leq 1$. Determine and plot

$Y_1(j\omega)$, $Y_2(j\omega)$, $Z_1(j\omega)$, $Z_2(j\omega)$, $w_1(t)$, and $w_2(t)$. $Y(j\omega) \leftrightarrow y(t)$ and $Z(j\omega) \leftrightarrow z(t)$.

8-33. Consider the following system:



The block labeled LP is an ideal low-pass filter with bandwidth $-2 \le \omega \le 2$ The time function $f(t)$ has a Fourier transform $F(j\omega) = 10\{u(\omega + 2) - u(\omega - 2)\}$. Determine and plot $Z(j\omega)$, $Q(j\omega)$, $P(j\omega)$, and $q(t)$.

$$Z(j\omega) \leftrightarrow z(t), \qquad Q(j\omega) \leftrightarrow q(t), \qquad P(j\omega) \leftrightarrow p(t)$$

8-34. A low-pass filter has the system function:

$$H(j\omega) = \frac{10(10 + j\omega)}{(5 + j\omega)(20 + j\omega)}$$

Determine the cutoff frequency ω_0. Then normalize the frequency by replacing ω by ω/ω_0. Convert this normalized low-pass filter into a band-pass filter when $\omega_2 = 50$ and $\omega_1 = 40$. ω_2 and ω_1 are respectively the upper and lower cutoff frequencies. Plot the magnitude of this band-pass filter.

Chapter 9

The Discrete Fourier Transform and the Fast Fourier Transform

INTRODUCTION

In Section 8-6 we illustrated the essentials of the Fourier analysis by considering a number of applications. Most of these applications were from the communications area. Another area of engineering science that is becoming increasingly important is that of signal processing. Within this field, the digital or discrete Fourier transform is beginning to play a large role. Real signals, like radar tracks, which are often processed with the Fourier transform in order to reveal their spectral content, are typically measured at discrete points in time, resulting in discrete time signals, $f(n)$. These discrete or discretized time signals call for some kind of discrete Fourier transform (DFT).

Thus the need for a DFT arises from discrete signals. From a slightly different point of view, let us recall the definitions:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \tag{9-1}$$

and

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega \tag{9-2}$$

The numerical computation of these integrals using digital computer processing requires that we take the continuous signals $f(t)$ and $F(j\omega)$ and discretize them. Also, we replace the integrals by finite summations. These manipulations lead directly to a discrete Fourier transform and an inverse discrete Fourier transform (IDFT). After a discussion of the DFT and the IDFT, we consider the problems of aliasing and leakage and the technique of windowing, all of which are relevant to the DFT. Then we investigate some of the DFT properties, after which we examine some efficient ways to compute the DFT and the IDFT.