Chapter

The Fourier Transfo

INTRODUCTION

This chapter will develop the Fourier transform and discuss a numbe properties and applications. The material is treated in a manner slightly di from previous transform chapters: instead of starting with the Fourier tra definition, Fourier series analysis will provide our point of departu conceptually and intuitively appealing ideas of Fourier series analy probably familiar to most readers. However, we begin from basics in Sect and then briefly consider the generalized Fourier series in Section 8- Fourier transform is developed in Section 8-3 and early emphasis will be on deriving a number of Fourier transform pairs, that is, the time functio and its corresponding transform $F(j\omega)$. In Section 8-4 a number of F transform properties will be developed, including the time-shift property a convolution property. The last section in this chapter examines a num Fourier transform applications, including filters, modulation, and multip For the most part, the signals considered in this chapter will be conti Chapter 9 deals with the Fourier analysis of discrete signals.

Jean Baptiste Joseph Fourier (1768–1830) was a French mathem and physicist who did extensive study of heat conduction. He developed now called the **Fourier series analysis** to be applied to the solution of differential equations that arose out of his heat conduction studies. The some heated controversy surrounding his publications, however, because not able to prove in a general fashion that his infinite series of sines and c actually converged to the function they were supposed to represent. No one prove it at first. It took about a hundred years and the invention of the Leb integral to do the job. For an overview of Fourier's life and times see Oppen

Willsky, and Young, pp. 162–168. One rather strange thing in Fourier's life might be explained by the time he spent in Egypt and his interests in Egyptology, as well as his intense involvement in heat studies: He believed dry desert heat to be ideal for health and lived the latter part of his life wrapped like a mummy in overheated rooms. Genius is permitted its eccentricities!

To motivate this Fourier analysis project, imagine that we have an $f(t)$ sampled to yield $f(n)$. Then, recalling Equation 1-15, we can write:

$$f(n) = \sum_{k=-\infty}^{\infty} f(k)\delta(k-n) \tag{8-1}$$

Thus any $f(t)$ can be approximated by a sum of unit pulse functions. The $f(k)$ values are constants that represent samples of the original $f(t)$. If $f(n)$ is the input to a linear system with unit pulse response $h(n)$, then the output of that system—as we saw in Chapter 2—was $g(n)$, where:

$$g(n) = f(n) * h(n) \tag{8-2}$$

The response of a linear system to a signal represented by unit pulses requires convolution. Determining the input signal representation is usually easy compared to performing the convolution required in Equation 8-2. Not only is convolution difficult to perform, but also the convolution operation calls for the unit pulse response function which may be difficult to determine.

Different kinds of representations of signals, however, might permit simpler response calculations. We know, for example, from sinusoidal steady-state circuit analysis that representing signals as sinusoids is computationally attractive. If the input to an *RLC* circuit is a sinusoid, then the output sinusoid of such a circuit has the same frequency as the input and differs only in amplitude and phase. Representing signals as sinusoids or as sums of sinusoids has certain advantages over representing signals as sums of unit pulses. These signals or functions "in terms of which" a given function is to be represented are called **basis functions.** We consider the employment of basis functions in Section 8-2. Note now only that unit pulse functions and sinusoids are particularly useful basis functions. Sinusoids or complex exponentials will be the type of basis function that is most commonly encountered throughout the rest of this Fourier transform chapter.

8-1 THE TRIGONOMETRIC FOURIER SERIES

Fourier's genius developed the insight that any periodic $f(t)$ can be expressed as a sum of sinusoids. A periodic function, $f(t)$, is one such that $f(t + nT) = f(t)$ for all integers n. T is the period.

Then

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \operatorname{Cos} n\omega_0 t + b_n \operatorname{Sin} n\omega_0 t \tag{8-3}$$

where

$$a_n = \frac{2}{T}\int_T f(\xi) \operatorname{Cos} n\omega_0 \xi \, d\xi \tag{8-4}$$

$$b_n = \frac{2}{T}\int_T f(\xi)\ \text{Sin}\ n\omega_0\xi\ d\xi \qquad (8\text{-}$$

for $n = 1, 2, \ldots$ The a_0 term is the dc component or average value of $f(t)$:

$$a_0 = \frac{1}{T}\int_T f(\xi)\ d\xi \qquad (8\text{-}$$

In these equations the integration symbol with "T" subscript implies that we c integrate over *any* period. Also, the ω_0 term which is called the fundamen angular frequency is $\omega_0 = 2\pi/T$. Note that all the sinusoids appearing Equation 8-3 have frequencies that are integer multiples of the fundamen This Fourier series representation in Equation 8-3 is known as the **trigonome Fourier series.**

EXAMPLE 8-1

Determine the trigonometric Fourier series expansion of $f(t)$ in Fig 8-1.

Solution

$T = 2$, $\omega_0 = \pi$, and $f(t) = t$ in the region $0 \le t \le 1$.

$$a_0 = \frac{1}{2}\int_0^1 \xi\ d\xi = \frac{1}{2}\frac{\xi^2}{2}\bigg|_0^1 = \frac{1}{4}$$

$$a_n = \int_0^1 \xi\ \text{Cos}\ n\pi\xi\ d\xi = \frac{(-1)^n - 1}{n^2\pi^2}$$

and

$$b_n = \int_0^1 \xi\ \text{Sin}\ n\pi\xi\ d\xi = \frac{1}{n\pi}(-1)^{n+1}$$

Therefore

$$f(t) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{(n\pi)^2}\ \text{Cos}\ n\pi t + \frac{(-1)^{n+1}}{n\pi}\ \text{Sin}\ n\pi\right.$$

EXAMPLE 8-2

Determine the trigonometric Fourier series expansion of $f(t)$ in Fig 8-2.

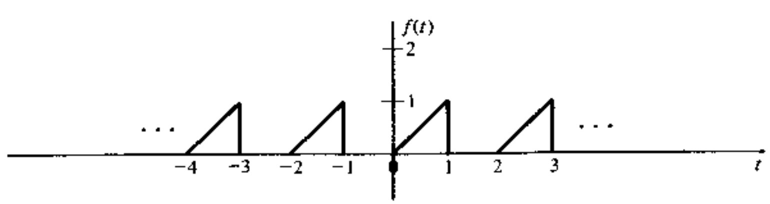


Figure 8-1 Periodic $f(t)$ of Example 8-1.

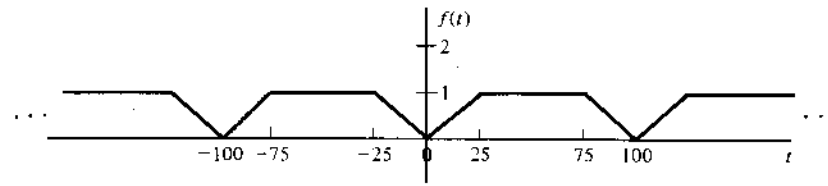


Figure 8-2 Periodic $f(t)$ of Example 8-2.

Solution

$T = 100$, $\omega_0 = \pi/50$. By inspection $a_0 = \frac{3}{4}$.

$$b_n = \frac{1}{50}\int_0^{25}\frac{\xi}{25}\ \text{Sin}\ \frac{n\pi\xi}{50}\ d\xi + \frac{1}{50}\int_{25}^{75}(1)\ \text{Sin}\ \frac{n\pi\xi}{50}\ d\xi + \frac{1}{50}\int_{75}^{100}\left(4 - \frac{\xi}{25}\right)\text{Sin}\ \frac{n\pi\xi}{50}\ d\xi$$

$$= \frac{1}{50}\left[\int_0^{25}\frac{\xi}{25}\ \text{Sin}\ \frac{n\pi\xi}{50}\ d\xi - \int_{75}^{100}\frac{\xi}{25}\ \text{Sin}\ \frac{n\pi\xi}{50}\ d\xi\right] + \frac{1}{50}\int_{25}^{75}\text{Sin}\ \frac{n\pi\xi}{50}\ d\xi + \frac{4}{50}\int_{75}^{100}\text{Sin}\ \frac{n\pi\xi}{50}\ d\xi = 0$$

and from an equation similar to the equation for b_n, we get:

$$a_n = \frac{4(\text{Cos}\ n\pi/2 - 1)}{(n\pi)^2}$$

Therefore

$$f(t) = \frac{3}{4} - \frac{4}{\pi^2}\left(\text{Cos}\ \frac{\pi}{50}t + \frac{1}{2}\text{Cos}\ \frac{2\pi}{50}t + \frac{1}{9}\text{Cos}\ \frac{3\pi}{50}t + \cdots\right)$$

Strictly speaking, the functions $f(t)$ that we are representing must be well behaved in order that the series expressing them will converge. This means that the periodic $f(t)$ of interest needs to satisfy what are called the **Dirichlet conditions:** $f(t)$ must have at most a finite number of maxima and minima and finite discontinuities in one period, and $f(t)$ must be absolutely integrable over one period. Absolute integrability means that:

$$\int_{-T/2}^{T/2} |f(t)|\ dt < \infty \qquad (8\text{-}7)$$

If these conditions are satisfied, then the Fourier series representation of $f(t)$ converges to the actual $f(t)$. The Dirichlet conditions are sufficient conditions; that is, it is not necessarily true that if the series converges then the conditions are satisfied. Fortunately, most engineering applications employ functions that do satisfy the Dirichlet conditions.

Now, if a Fourier series representation of a periodic signal is obtained and this signal is used as an input to a linear system, then what is the forced output of the system? In order to deal with this situation most effectively, we need to express our periodic signals in a way that combines the sine and cosine terms in the original expansion into a single term with a phase shift. We can write a cosinusoidal Fourier series as:

$$f(t) = a_0 + \sum_{n=1}^{\infty} \tilde{c}_n \operatorname{Cos}(n\omega_0 t + \theta_n) \quad (8$$

where $\tilde{c}_n = \sqrt{a_n^2 + b_n^2}$ and $\theta_n = -\operatorname{Tan}^{-1} b_n/a_n$ (8

Now let $x(t)$ be a periodic input to a system that has a transfer function $H($ and let $y(t)$ be the system output.

Then
$$x(t) = a_0 + \sum_{n=1}^{\infty} \tilde{c}_n \operatorname{Cos}(n\omega_0 t + \theta_n) \quad (8\text{-}$$

$$H(jn\omega_0) = H(j\omega)|_{\omega=n\omega_0} = |H(jn\omega_0)| \angle \arg H(jn\omega_0) \quad (8$$

and
$$y(t) = a_0 H(0) + \sum_{n=1}^{\infty} \tilde{c}_n |H(jn\omega_0)| \operatorname{Cos}(n\omega_0 t + \theta_n + \arg H(jn\omega_0)) \quad (8$$

EXAMPLE 8-3

Consider the response $y(t)$ of the system $H(j\omega) = j\omega/(j\omega + 2)$ when input $x(t)$ is a periodic signal of period $T = 4$

and
$$x(t) = 0, \quad -2 \le t \le -1$$
$$= \operatorname{Cos}\frac{\pi}{2}t, \quad -1 \le t \le 1$$
$$= 0, \quad 1 \le t \le 2$$

This is actually a half-wave rectified cosine function, which appears Figure 8-3. Determine the dc term, the first harmonic (fundamental), the second harmonic in the response $y(t)$.

Solution

$$x(t) = \frac{1}{\pi} + \frac{1}{2}\operatorname{Cos}\frac{\pi t}{2} - \sum_{n=1}^{\infty} \frac{2(-1)^n}{\pi(4n-1)} \operatorname{Cos} n\pi t$$

follows from a straightforward (but very tedious) application of Equat 8-4 through 8-6. Now $\omega_0 = 2\pi/T = \pi/2$. The a_0 or dc term in $x(t)$ is The first harmonic term in $x(t)$ is $\frac{1}{2}\operatorname{Cos}\pi t/2$. The second harmonic ter

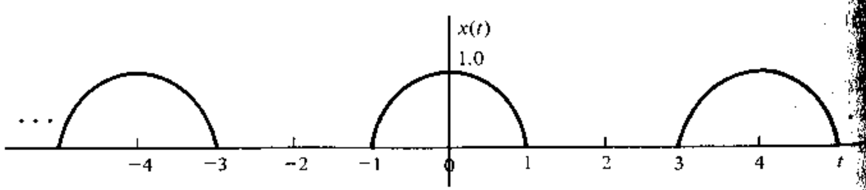


Figure 8-3 Periodic $x(t)$ of Example 8-3.

$x(t)$ is the first term in the summation: $-2(-1)/\pi(4 - 1) \operatorname{Cos} \pi t = (2/3\pi) \operatorname{Cos} \pi t$. Now from the system transfer function:

$$H(jn\omega_0) = H\left(jn\frac{\pi}{2}\right) = \frac{jn\pi/2}{jn\pi/2 + 2} = \frac{n\pi/2}{\sqrt{(n\pi/2)^2 + 4}} \angle\left(90^\circ - \operatorname{Tan}^{-1}\frac{n\pi}{4}\right)$$

Therefore
$$H(0) = 0, \quad H\left(j\frac{\pi}{2}\right) = \frac{\pi}{\sqrt{\pi^2 + 16}} \angle\left(90^\circ - \operatorname{Tan}^{-1}\frac{\pi}{4}\right) = 0.618 \angle 51.85^\circ$$

$$H(j\pi) = \frac{\pi}{\sqrt{\pi^2 + 4}} \angle\left(90^\circ - \operatorname{Tan}^{-1}\frac{\pi}{2}\right) = 0.844 \angle\ 32.48^\circ$$

and in the output, we get:

$$\text{dc} = a_0 H(0) = 0$$

$$\text{first harmonic} = 0.618(0.5)\operatorname{Cos}\left(\frac{\pi t}{2} + 51.85^\circ\right) = 0.309 \operatorname{Cos}\left(\frac{\pi t}{2} + 51.85^\circ\right)$$

$$\text{second harmonic} = 0.844\left(\frac{2}{3\pi}\right)\operatorname{Cos}(\pi t + 32.48^\circ) = 0.179 \operatorname{Cos}(\pi t + 32.48^\circ)$$

The concepts of evenness and oddness are useful in the Fourier series theory. An even function $f_e(t)$ is one such that:

$$f_e(-t) = f_e(t) \quad (8\text{-}13)$$

An odd function $f_0(t)$ is one such that:

$$f_0(-t) = -f_0(t) \quad (8\text{-}14)$$

An interesting fact is that *any* $f(t)$ can be written:

$$f(t) = f_e(t) + f_0(t) \quad (8\text{-}15)$$

where
$$f_e(t) = \frac{f(t) + f(-t)}{2} \quad (8\text{-}16)$$

and
$$f_0(t) = \frac{f(t) - f(-t)}{2} \quad (8\text{-}17)$$

EXAMPLE 8-4

Determine and plot $f_e(t)$ and $f_0(t)$ if $f(t) = u(t)$.

Solution

$$f_e(t) = \frac{u(t) + u(-t)}{2} = \frac{1}{2}, \quad \text{for all } t, \text{ except } t = 0$$

$$f_0(t) = \frac{u(t) - u(-t)}{2} = -\frac{1}{2}, \quad \text{for } t < 0$$

$$= \frac{1}{2}, \quad \text{for } t > 0$$

The only problem here is the value of these functions at $t = 0$. Since $u(t)$ defined to be 1.0 for $t \geq 0$, then $u(-t) = 1$ for $t \leq 0$.

Therefore $\quad f_e(0) = 1 \quad \text{and} \quad f_0(0) = 0$

The functions $f_e(t)$ and $f_0(t)$ are plotted in Figure 8-4.

The evenness and oddness of certain functions can be used to simplify calculations for a_n and b_n required in the trigonometric Fourier series. Gi some $f(t)$, if this $f(t)$ is *odd*,

then $\quad a_n = 0, \quad \text{for } n = 0, 1, 2, \cdots$ (8-

$$b_n = \frac{4}{T}\int_{T/2} f(\xi) \text{ Sin } n\omega_0\xi \, d\xi, \quad \text{for } n = 1, 2, \cdots$$

If some given $f(t)$ is *even*,

then $\quad a_0 = \frac{2}{T}\int_{T/2} f(\xi)\, d\xi$ (8

$$a_n = \frac{4}{T}\int_{T/2} f(\xi) \text{ Cos } n\omega_0\xi \, d\xi, \quad \text{for } n = 1, 2, \cdots$$

$$b_n = 0, \quad \text{for } n = 1, 2, \cdots$$

These results follow from the fact that $\int_T = 2\int_{T/2}$ if the integrand is *even* and $\int$ 0 if the integrand is *odd*.

EXAMPLE 8-5

Determine the trigonometric Fourier series expansion for the $f(t)$ give Figure 8-5.

Solution

$$T = 3, \; \omega_0 = \frac{2\pi}{T} = \frac{2}{3}\pi$$

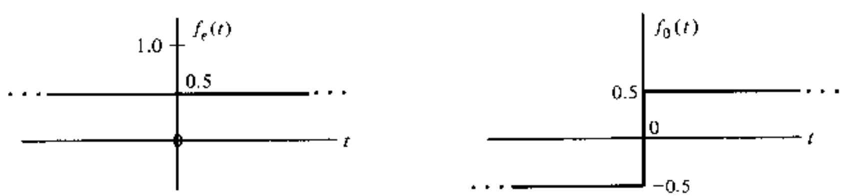


Figure 8-4 $f_e(t)$ and $f_0(t)$ of Example 8-4.

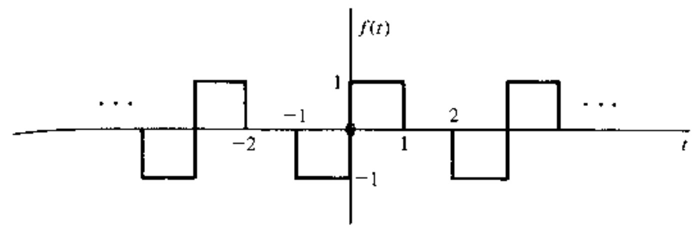


Figure 8-5 Periodic $f(t)$ of Example 8-5.

By inspection $f(t)$ is odd.

Therefore
$$a_n = 0, \; b_n = \frac{4}{3}\int_0^1 (1) \text{ Sin } n\frac{2\pi}{3}\xi\, d\xi$$

$$= -\frac{4}{3}\left(\frac{3}{2\pi n}\right)\left[\text{Cos } \frac{2\pi n}{3} - 1\right]$$

thus
$$f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1 - \text{Cos } \frac{2\pi n}{3}\right) \text{Sin } \frac{2\pi n}{3}t$$

8-2 GENERALIZED FOURIER SERIES

The trigonometric Fourier series represents a periodic function as an infinite series of sinusoids. In a more general sense, any function can be expressed as an infinite series of other functions or can at least be approximated by a finite series of other functions. Call these "other functions" **basis functions:** $\phi_0(t)$, $\phi_1(t)$, $\phi_2(t)$, ... Assume we can approximate $f(t)$ over a certain range $t_1 \leq t \leq t_2$ with the function $\hat{f}(t)$:

$$\hat{f}(t) = \sum_{i=0}^{N} \alpha_i\phi_i(t) \tag{8-20}$$

where N may be infinity. Generally speaking, the more terms we take, the closer $\hat{f}(t)$ will be to $f(t)$. We will limit the possible spread of the basis functions by demanding that they have certain properties that will result in elegant formulations. Insist that the basis functions be **orthogonal** over the range $t_1 \leq t \leq t_2$; that is:

$$\int_{t_1}^{t_2} \phi_i(t)\phi_j^*(t)\, dt = 0, \quad i \neq j$$
$$= \lambda_i, \quad i = j \tag{8–21}$$

The asterisk (*) notation indicates complex conjugation and must be employed if the basis functions are complex functions of time. There is a special case of Equation 8-21 where $\lambda_i = 1$ for all i. In this case the basis functions are said to be **orthonormal.** Referring to Equation 8-20, assuming the ϕ_i's are known, the problem is to determine the proper α_i values. To do so, multiply both sides of

Equation 8-20 by $\phi_j^*(t)$ and integrate over $[t_1, t_2]$ to get:

$$\int_{t_1}^{t_2} \hat{f}(t)\phi_j^*(t)\,dt = \sum_{i=0}^{N} \alpha_i \int_{t_1}^{t_2} \phi_i(t)\phi_j^*(t)\,dt \qquad (8\text{-}2$$

But from Equation 8-21, the integral on the right is zero unless $i = j$. Thus on the term for $i = j$ in the summation will remain and when $i = j$ the integral on t right becomes λ_j. We get:

$$\int_{t_1}^{t_2} \hat{f}(t)\phi_j^*(t)\,dt = \alpha_j\lambda_j \qquad (8\text{-}2$$

Unfortunately, this integral requires us to use $\hat{f}(t)$ which is what we are trying determine. But $\hat{f}(t)$ is supposed to be "close" to $f(t)$. Substituting $f(t)$ for $\hat{f}$ we obtain what are called the **generalized Fourier series coefficients** (chang the index j to i).

$$\alpha_i = \frac{1}{\lambda_i}\int_{t_1}^{t_2} f(t)\phi_i^*(t)\,dt \qquad (8\text{-}$$

These α_i values substituted back into Equation 8-20 give $\hat{f}(t)$ as the b approximation to $f(t)$, best in the sense of what is called the "minimum m square error." The function $\hat{f}(t)$ is supposed to be a good approximation to f The mean square error (MSE) is a measure of how good "good" is:

$$\text{MSE} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \left| f(t) - \sum_{i=0}^{N} \alpha_i\phi_i(t) \right|^2 dt \qquad (8\text{-}$$

Intuitively, as N gets larger, in most cases the MSE gets smaller. If $\lim_{N\to\infty}$ $= f(t)$, then the basis functions are said to be **complete.** In this case the MSE and by equating Equation 8-25 to zero we can derive what is called **Parse relation:**

$$\int_{t_1}^{t_2} f^2(t)\,dt = \sum_{i=0}^{\infty} |\alpha_i|^2\lambda_i \qquad (8\text{-}$$

EXAMPLE 8-6

Demonstrate Parseval's relation.

Solution. Carry out Equation 8-25:

$$\text{MSE} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\left[f^2(t) - f(t)\sum_{i=0}^{N}\alpha_i\phi_i(t) - f(t)\sum_{i=0}^{N}\alpha_i^*\phi_i^*(t) + \sum_{j=0}^{N}\alpha_j\phi_j(t)\sum_{i=0}^{N}\alpha_i^*\phi_i^*(t)\right]$$

since for complex numbers $|z|^2 = zz^*$. From Equation 8-21, we get the l term:

$$\frac{1}{t_2 - t_1}\sum_{i=0}^{N}\lambda_i|\alpha_i|^2$$

But the two middle terms become:

$$\frac{-2}{t_2 - t_1}\sum_{i=0}^{N}|\alpha_i|^2\lambda_i$$

from Equation 8-24. This assumes that λ_i and $f(t)$ take on only real values, whereas ϕ_i and α_i may be complex.

Hence $$\text{MSE} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f^2(t)\,dt - \frac{1}{t_2 - t_1}\Sigma|\alpha_i|^2\lambda_i$$

and if MSE $\to 0$,

then $$\int_{t_1}^{t_2} f^2(t)\,dt \to \sum_{i=0}^{N}|\alpha_i|^2\lambda_i$$

which becomes, as $N \to \infty$:

$$\int_{t_1}^{t_2} f^2(t)\,dt = \sum_{i=0}^{\infty}|\alpha_i|^2\lambda_i.$$

Parseval's relation is an equation relating energy in the actual signal to energy contained in the signal's representation. If we can make the calculation on the left side of the equation—call it E—then it is often desired to compare E, not to the infinite sum on the right side, but to the sum of a finite number of these terms. For example, we might want to take enough terms in the summation such that the right side is at least 95% of E. Or we can calculate $(E - \Sigma|\alpha_i|^2\lambda_i)/E$ as a relative energy error and try to minimize this term. In the real world we typically approximate a given $f(t)$ by as few basis functions as possible. The level of accuracy needed in a given problem is usually determined by the overall problem context.

EXAMPLE 8-7

Assume we are given the basis functions ϕ_1, ϕ_2, and ϕ_3 of Figure 8-6. Approximate:

$$f(t) = t, \quad 0 \le t \le 1$$
$$= 0, \quad \text{otherwise}$$

in terms of ϕ_1, ϕ_2, and ϕ_3 as $\hat{f}(t) = \alpha_1\phi_1 + \alpha_2\phi_2 + \alpha_3\phi_3$ and determine the relative energy error.

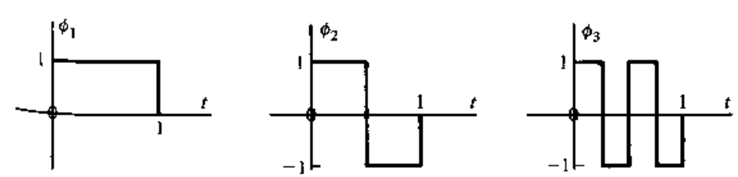


Figure 8-6 Basis functions for Example 8-7.

Solution. We need to check our basis functions using Equation 8-21. In total, we have six integrals to compute. Since $\int_0^1 \phi_i\phi_i\, dt = 1$ for $i = 1, 2, 3$, and $\int_0^1 \phi_i\phi_j\, dt = 0$ for $i \neq j$, we can conclude that the basis functions are orthonormal and $\lambda_1 = \lambda_2 = \lambda_3 = 1$. From Equation 8-24 we have:

$$\alpha_1 = \int_0^1 t\, dt = 0.5,$$

$$\alpha_2 = \int_0^{0.5} t\, dt - \int_{0.5}^1 t\, dt = -0.25$$

and
$$\alpha_3 = \int_0^{0.25} t\, dt - \int_{0.25}^{0.5} t\, dt + \int_{0.5}^{0.75} t\, dt - \int_{0.75}^1 t\, dt = -0.125$$

Therefore $\hat{f}(t) = 0.5\,\phi_1 - 0.25\,\phi_2 - 0.125\,\phi_3$

A comparison between $f(t)$ and $\hat{f}(t)$ is shown in Figure 8-7.

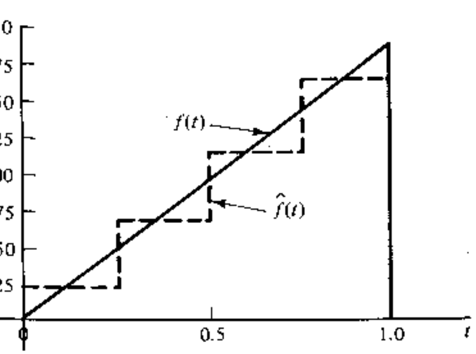
Also,
$$E = \int_0^1 f^2\, dt = \int_0^1 t^2\, dt = \frac{1}{3}t^3\Big|_0^1 = 0.333$$

and
$$\Sigma\, \alpha_i^2\lambda_i = (0.5)^2 + (0.25)^2 + (0.125)^2 = 0.328$$

Therefore
$$\frac{E - \Sigma\,\alpha_i^2\lambda_i}{E} = \frac{0.333 - 0.328}{0.333} = 0.0146 \rightarrow 1.46\%$$

This small error implies that $\hat{f}(t)$ very closely approximates $f(t)$.

Now in these generalized Fourier Series representations we have assumed a finite time duration is of interest: $t_1 \le t \le t_2$. If we are dealing with periodic ...tions, then that duration can be considered to be one period of the periodic ...tion. Let us apply the generalized Fourier series ideas to the development of ... complex exponential Fourier series. Assume we are given $f(t)$ which is ...dic with period T. Let $\phi_n(t)$ be a *complete* set of basis functions: $\phi_n(t) =$ ... $n = 0, \pm1, \pm2, \ldots$, where $\omega_0 = 2\pi/T$.



...re 8-7 Comparison of $f(t)$ and $\hat{f}(t)$ for Example 8-7.

Then
$$f(t) = \hat{f}(t) = \sum_{n=-\infty}^{\infty} \alpha_n\phi_n(t) \tag{8-27}$$

The one-sided summation used in Equation 8-20 in this case becomes a two-sided summation. This is permissible because we have an infinite number of terms and the indexing is arbitrary. For purposes of symmetry, the two-sided format is convenient for the complex exponential Fourier series. Note that n instead of i is the index in Equation 8-27. This is because the basis functions here are complex and contain j terms and we want to avoid mixing i and j terms. Mathematicians use i where engineers typically use j. Also, the basis functions employed in this representation are orthogonal over $0 < t < T$ with $\lambda_n = T$ for all n.

EXAMPLE 8-8

Prove that the exponential Fourier series basis functions are orthogonal with $\lambda_n = T$ for all n over $0 \le t \le T$.

Solution

$$\int_{t_1}^{t_2} \phi_n(t)\phi_m{}^*(t)\, dt = \int_{t_0}^{t_0+T} e^{jn\omega_0 t}\, e^{-jm\omega_0 t}\, dt$$

$$= \frac{1}{j\omega_0(n-m)}\left[e^{j\omega_0(n-m)(t_0+T)} - e^{j\omega_0(n-m)t_0}\right]$$

but $n - m = k$ an integer and $e^{j\omega_0 k t_0}\, e^{j\omega_0 kT} - e^{j\omega_0 k t_0} = e^{j\omega_0 k t_0}\,(e^{j\omega_0 kT} - 1)$ and $\omega_0 kT = 2\pi k$ and $e^{j2\pi k} = 1$. Thus the term in square brackets $= 0$. Only when $n = m$ will things be otherwise. When $n = m$ we get:

$$\int_{t_1}^{t_2} \phi_n(t)\phi_m^*(t)\, dt = \int_{t_0}^{t_0+T} dt = T$$

Therefore
$$\int_{t_1}^{t_2} \phi_n(t)\phi_m^*(t)\, dt = T, \quad \text{if } n = m$$
$$= 0, \quad \text{otherwise}$$

and the exponential Fourier series has orthogonal basis functions with $\lambda_n = T$ for all n.

Now any periodic $f(t)$ can be written

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n\phi_n(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega_0 t} \tag{8-28}$$

In order to determine α_n, we use Equation 8-24:

$$\alpha_n = \frac{1}{T}\int_T f(t)e^{-jn\omega_0 t}\, dt \tag{8-29}$$

These are called **complex exponential Fourier series coefficients** and usually are written as c_n instead of α_n in order to distinguish this particular Fourier series.

The complex exponential Fourier series is closely related to the trigonometric Fourier series. The relationship between these representations can be made explicit by applying the Euler identities. We can write the complex exponential in Equation 8-28 as Cos $n\omega_0 t$ + j Sin $n\omega_0 t$. Then comparing Equation 8-28 to the representation in Equation 8-3, we can deduce the following:

$$a_n = c_n + c_{-n}$$

$$b_n = j(c_n - c_{-n})$$

$$c_n = \frac{a_n - jb_n}{2} \quad \text{and} \quad c_{-n} = \frac{a_n + jb_n}{2}, \qquad \text{for } n > 0$$

$$c_0 = a_0 \tag{8-30}$$

EXAMPLE 8-9

Determine the exponential Fourier series representation of the periodic $f(t)$ which $= e^t$, $0 \le t \le 1$, and which has $T = 1$. This $f(t)$ is sketched in Figure 8-8.

Solution

$$\omega_0 = 2\pi, \qquad c_n = \frac{1}{1}\int_0^1 e^t e^{-jn\omega_0 t}\,dt = \frac{1}{1 - jn\omega_0}(e^{1-jn\omega_0} - 1)$$

But $$e^{-jn\omega_0} = e^{-jn2\pi} = 1$$

and $$e^{1-jn\omega_0} = e^1 = 2.718$$

Therefore $$c_n = \frac{1.718}{1 - jn2\pi} \quad \text{and} \quad f(t) = 1.718\sum_{n=-\infty}^{\infty}\frac{e^{jn2\pi t}}{1 - jn2\pi}$$

$$= 1.718\sum_{n=-\infty}^{\infty}\frac{e^{jn\omega_0 t}}{1 - jn\omega_0}$$

Often, the relationship between $f(t)$ and c_n, such as the relationship between $f(t)$ and its Laplace transform or between $f(n)$ and its Z transform, is indicated by the double arrow notation: $f(t) \leftrightarrow c_n$. Note that c_n in the last example was a complex number. We can write:

$$c_n = |c_n| \angle\theta_n = |c(n\omega_0)| \angle\theta(n\omega_0) \tag{8-31}$$

If we plot $|c(n\omega_0)|$ versus $n\omega_0$ and $\theta(n\omega_0)$ versus $n\omega_0$, we have what is called the

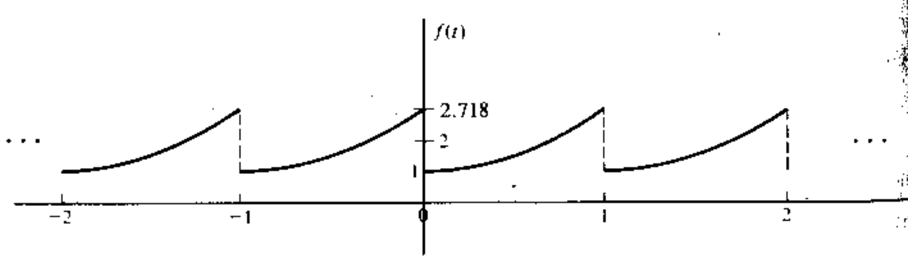


Figure 8-8 Periodic $f(t)$ of Example 8-9.

complex Fourier spectrum. Generally, these plots are points, discrete numbers at discrete values of $n\omega_0$. We can make the plots more dramatic by dropping the points to the $n\omega_0$ axis to form **line spectra,** typified by the plots in Figure 8-9 which represent the line spectra of the previous example. These lines indicate the spectral content of the signal. We normally plot the magnitude and phase line spectra as functions of $n\omega_0$ instead of just n because later, in the development of the Fourier transform, we will have ω as our independent variable and ω comes directly from $n\omega_0$.

Observation of Figure 8-9 reveals an interesting result: The magnitude spectrum is an *even* function of $n\omega_0$ and the phase spectrum is an *odd* function of $n\omega_0$. This is true for the $f(t)$ of Example 8-9, but is it always the case? To determine what must be the case for c_n to have an even magnitude spectrum and an odd phase spectrum, we take the complex conjugate of Equation 8-29, with α_n replaced by c_n:

$$c_n^* = \frac{1}{T}\int_T \{f(t)e^{-jn\omega_0 t}\,dt\}^*$$

$$= \frac{1}{T}\int_T \{f(t)e^{-jn\omega_0 t}\}^*\,dt$$

$$= \frac{1}{T}\int_T f^*(t)e^{jn\omega_0 t}\,dt \tag{8-32}$$

Now from Equation 8-31

$$c_n^* = \{|c(n\omega_0)| \angle\theta(n\omega_0)\}^* = \{|c(n\omega_0)|\,e^{j\theta(n\omega_0)}\}^*$$

$$= |c(n\omega_0)|\,e^{-j\theta(n\omega_0)} \tag{8-33}$$

Assume that the magnitude is an even function of $n\omega_0$,

then $$|c(-n\omega_0)| = |c(n\omega_0)|$$

Assume that the phase is an odd function of $n\omega_0$

then $$\theta(-n\omega_0) = -\theta(n\omega_0)$$

Therefore $$c_n^* = |c(-n\omega_0)|\,e^{j\theta(-n\omega_0)} = c_{-n} \tag{8-34}$$

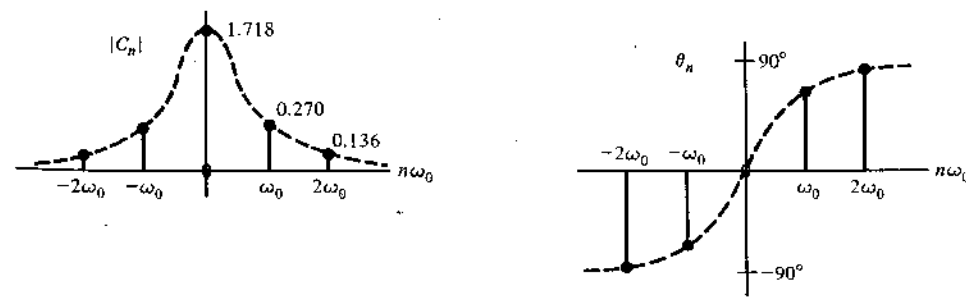


Figure 8-9 The magnitude and phase of c_n from Example 8-9.

But from the integral equation, Equation 8-29, we get:

$$c_{-n} = \frac{1}{T}\int_T f(t)\,e^{jn\omega_0 t}\,dt \tag{8-35}$$

and if this equals

$$c_n^* = \frac{1}{T}\int_T f^*(t)\,e^{jn\omega_0 t}\,dt$$

then $f^*(t) = f(t)$; that is, the time function whose Fourier series we are interested in must be a *real* function. This is the condition under which the magnitude and phase of c_n are respectively even and odd functions of $n\omega_0$.

Also, note that since, for real $f(t)$, $\theta(0) = 0$, c_0 is a real number. This should make intuitive sense because from Equation 8-30 we have $c_0 = a_0$. This is just the average or dc value of the given time function $f(t)$.

Now, as we have seen, the trigonometric and the exponential Fourier series are closely related. The trigonometric series very clearly displays the given periodic $f(t)$ as a dc term plus a sum of sinusoids. The exponential Fourier series, on the other hand, is a compact expression. That is its appeal, plus the fact that it leads very nicely into the Fourier transform which is considered in the next section. A point of confusion concerning the exponential Fourier series is often expressed in the question: How can a *real* $f(t)$ be represented by a summation of basis functions that are *complex*? The answer is that although the basis functions are complex, they appear in complex conjugate pairs that reduce to real sines and cosines.

The generalized Fourier series methods permit us to represent a given signal in terms of other signals that may be easier to handle. For periodic signals, the trigonometric, cosinusoidal, and exponential Fourier series methods provide useful representations that reveal the spectral content of the given signal. Intuitively, the $f(t)$ in Figure 8-8, for instance, is composed of a dc term plus a number of sinusoids. These functions can be generated from the c_n plots of Figure 8-9. Fourier methods applied to periodic signals, then, provide representations *and* reveal spectral content. The generalized Fourier series methods applied to nonperiodic signals, on the other hand, are used typically to provide alternative representations for a given signal. They are seldom concerned with spectral content. To reveal the spectral content of nonperiodic signals, we use the methods of Fourier transform analysis. In fact, in the next section we develop the magnitude and phase of the Fourier transform to show the spectral content of nonperiodic signals, just as the c_n terms stand out in the complex exponential Fourier series to represent the spectral content of periodic signals.

Before turning to the Fourier transform, let us consider one more Fourier series example.

EXAMPLE 8-10

Determine the complex exponential Fourier series coefficients for the $f(t)$ represented in Figure 8-10. Then consider the effect of shifting $f(t)$ $d/2$ units to the right.

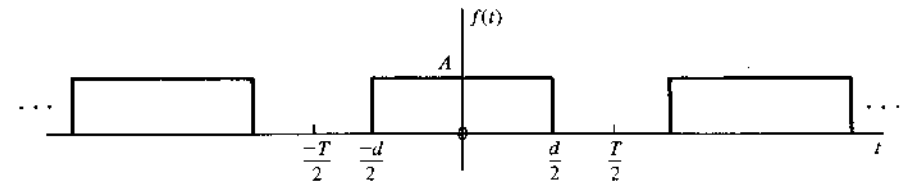


Figure 8-10 $f(t)$ for Example 8-10.

Solution

$$c_n = \frac{1}{T}\int_T f(t)\,e^{-jn\omega_0 t}\,dt$$

which becomes:

$$c_n = \frac{1}{T}\int_{-d/2}^{d/2} Ae^{-jn\omega_0 t}\,dt = \frac{A}{T}\frac{1}{(-jn\omega_0)}\left(e^{-jn\omega_0(d/2)} - e^{jn\omega_0(d/2)}\right)$$

$$= \frac{2A}{Tn\omega_0}\,\mathrm{Sin}\left(n\omega_0\frac{d}{2}\right) = \frac{Ad}{T}\frac{\mathrm{Sin}\,(n\omega_0(d/2))}{(n\omega_0 d/2)}$$

Now recall from Chapter 1 that Sinc (t) = Sin $\pi t/\pi t$.

Therefore
$$c_n = \frac{Ad}{T}\,\mathrm{Sinc}\left(\frac{n\omega_0 d}{2\pi}\right) = \frac{Ad}{T}\,\mathrm{Sinc}\left(\frac{nd}{T}\right)$$

For purposes of illustration we plot $|c_n|$ versus $n\omega_0$ in Figure 8-11 in the case where $A = 10$, $T = 10$, and $d = 2$. Note that the envelope of this curve is the familiar Sinc (x) pattern. Now if we shift $f(t)$ to the right by $d/2$ units, we can write:

$$c_n = \frac{1}{T}\int_0^d Ae^{-jn\omega_0 t}\,dt = \frac{A}{T}\frac{1}{(-jn\omega_0)}\left(e^{-jn\omega_0 d} - 1\right)$$

$$= \frac{A}{T}\frac{e^{-jn\omega_0(d/2)}}{(-jn\omega_0)}\left(e^{-jn\omega_0(d/2)} - e^{jn\omega_0(d/2)}\right) = e^{-jn\omega_0(d/2)}\frac{Ad}{T}\,\mathrm{Sinc}\left(\frac{n\omega_0 d}{2\pi}\right)$$

which is the same as c_n for the unshifted function except for the phase term $\angle -n\omega_0 d/2$. Therefore the magnitude of this new c_n will be the same as before and only the phase will be changed. This result, in fact, is very

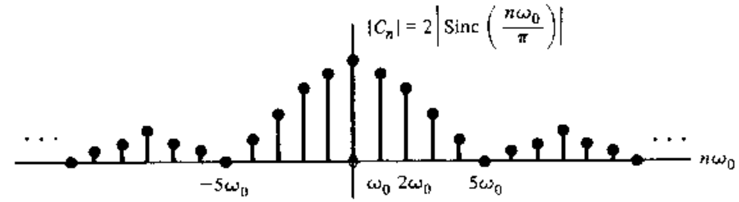


Figure 8-11 Plot of $|c_n|$ versus $n\omega_0$ for Example 8-10.

general and can be stated as follows: if $f(t) \leftrightarrow c_n$, then $f(t - t_0) \leftrightarrow e^{-jn\omega_0 t_0} c_n$. To prove it, let $f(t - t_0) \leftrightarrow \hat{c}_n$.

thus

$$\hat{c}_n = \frac{1}{T}\int_0^T f(t - t_0)\, e^{-jn\omega_0 t}\, dt$$

Let $\lambda = t - t_0$, $d\lambda = dt$.

then

$$\hat{c}_n = \frac{1}{T}\int_{-t_0}^{T-t_0} f(\lambda)\, e^{-jn\omega_0(\lambda + t_0)}\, d\lambda = e^{-jn\omega_0 t_0} c_n$$

Drill Set: Fourier Series

1. Prove that $\{\phi_n\} = \{\text{Sin } nt, \text{Cos } nt, n = 1, 2, \ldots\}$ constitute an orthogonal set of basis functions over the range $0 < t < 2\pi$.
2. $\phi_1(t) = \frac{1}{3}t$ and $\phi_2(t) = d_1 t^2 + d_2$ are known to be a pair of orthonormal basis functions on the interval $0 < t < t_1$. Find d_1, d_2, and t_1. Determine α_1 and α_2 where $\hat{f}(t) = \alpha_1\phi_1 + \alpha_2\phi_2$ and the function we want $\hat{f}(t)$ to approximate is the pulse $f(t) = u(t) - u(t - t_1)$.
3. Expand $f(t) = \text{Sin}^2\, 2\pi t\, \text{Cos}\, \pi t$ into a trigonometric Fourier series and into a complex exponential Fourier series. Plot c_n.
4. Determine and sketch the even and odd components of

$$f(t) = e^{-t}\,\text{Cos}\, t, \qquad t > 0$$
$$= 0, \qquad t < 0$$

5. Consider the periodic impulse train $f(t) = \Sigma_{n=-\infty}^{\infty}\, \delta(t + nT)$. Determine and plot the exponential Fourier series coefficients. How is c_n in this case unlike c_n terms in previous examples?
6. Let

$$f(t) = 10, \qquad 0 \le t \le 1$$
$$= 0, \qquad 1 \le t \le 2$$

be a periodic function with period $T = 2$. Assume $f(t) \approx k_1 + k_2\, \text{Sin}\, \omega_0 t + k_3\, \text{Sin}\, 3\omega_0 t$. Determine k_1, k_2, and k_3.

8-3 THE FOURIER TRANSFORM

Assume we have $f(t)$ in the range $-d/2 < t < d/2$. Outside this range $f(t) = 0$. Now the complex exponential Fourier series of Equation 8-28 can be used to describe $f(t)$ within the given range, but outside this range Equation 8-28 would not describe $f(t)$—since its true value is zero—but would instead describe a periodic extension of $f(t)$ in the range $-d/2 < t < d/2$. Assume that this periodic extension has a period T and that $d < T$. As an example, a glance at the periodic

$f(t)$ represented in Figure 8-10 might be helpful. As T gets larger and larger, the given $f(t)$ is more and more accurately represented by the right-hand side of Equation 8-28. As $T \to \infty$, Equation 8-28, in theory, exactly represents the given $f(t)$ which is nonzero for $-d/2 < t < d/2$ and equal to zero for t outside this range. For example, the $f(t)$ in Figure 8-10 would have only the middle pulse remaining as $T \to \infty$. This new function, instead of being considered a periodic function with infinite period, will be considered a nonperiodic function.

In order to formalize this development, let $\alpha_n = c_n$ in Equation 8-29 and plug it into Equation 8-28 to get:

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_T f(t)e^{-jn\omega_0 t}\, dt\; e^{jn\omega_0 t} \tag{8-36}$$

Now the spacing between harmonics, as illustrated in Figure 8-9, is just ω_0. But $\omega_0 = 2\pi/T$. In the limit as $T \to \infty$, ω_0 becomes infinitesimal; call it $d\omega$. Also, $n\omega_0$ becomes a continuous variable; call it ω. In addition, the summation becomes an integral. We can summarize the changes made in Equation 8-36 as $T \to \infty$:

$$\int_T \to \int_{-\infty}^{\infty}$$
$$\omega_0 \to d\omega$$
$$n\omega_0 \to \omega$$
$$\sum_{-\infty}^{\infty} \to \int_{-\infty}^{\infty}$$
$$\frac{1}{T} = \frac{\omega_0}{2\pi} \to \frac{d\omega}{2\pi} \tag{8-37}$$

Incorporating these changes in Equation 8-36, we obtain:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt\right] e^{j\omega t}\, d\omega \tag{8-38}$$

The term in the brackets is called the **Fourier transform of $f(t)$** and is indicated by $F(j\omega)$.

Thus

$$F(j\omega) = \text{FT}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt \tag{8-39}$$

Then the inverse Fourier transform is written

$$f(t) = \text{IFT}\{F(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\, d\omega \tag{8-40}$$

Equations 8-39 and 8-40 constitute what are called Fourier transform pairs and can be represented, like other transform pairs, as follows:

$$f(t) \leftrightarrow F(j\omega)$$

For the most part, corresponding to $f(t)$ there is a unique $F(j\omega)$ and corresponding to $F(j\omega)$ there is a unique $f(t)$. To get one from the other, we use an integral operator. The Fourier transform transforms a time-domain function into the frequency domain. The inverse Fourier transform transforms a frequency-domain function into the time domain. The frequency domain is indicated by ω, the radian frequency, which has units of radians/second. The more "natural" frequency domain is indicated by f, which has units of hertz. Of course, $f = \omega/2\pi$. Some texts will use the notation $F(f)$ or $F(\omega)$ to indicate the Fourier transform. These are particularly popular in texts whose focus is communication theory. For our purposes, however, $F(j\omega)$ will be a more useful notation because of our concern with the Laplace transform, $F(s)$. Often these transforms will be identical if we let $s = j\omega$ in the Laplace transform.

EXAMPLE 8-11

Determine the Fourier transform of the $f(t)$ indicated in Figure 8-10 when $T \to \infty$.

Solution

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt = \int_{-d/2}^{d/2} Ae^{-j\omega t}\,dt$$

$$= \frac{A}{-j\omega}\{e^{-j\omega d/2} - e^{j\omega d/2}\} = \frac{2A}{\omega}\text{Sin}\left(\frac{\omega d}{2}\right) = Ad\,\text{Sinc}\left(\frac{\omega d}{2\pi}\right)$$

Letting $A = 10$ and $d = 2$, we plot $F(j\omega)$ in Figure 8-12 as a familiar Sinc (x) curve.

Now in view of the results of this example, we cannot help noticing that there is a striking resemblance between this $F(j\omega)$ and the c_n from the previous example. From Example 8-10 we had $c_n = (Ad/T)$ Sinc $(n\omega_0 d/2\pi)$. Letting $n\omega_0 = \omega$ and multiplying c_n by T, we obtain the result of Example 8-11; that is, $F(j\omega) = Ad\,\text{Sinc}\,(\pi d/2\pi)$. This is a very general result. Imagine we have c_n for a periodic $f(t)$. Let

$$\tilde{f}(t) = f(t), \qquad \frac{-T}{2} \le t \le \frac{T}{2}$$

and assume $\tilde{f}(t)$ is zero outside this range.

Let $F(j\omega)$ be the Fourier transform of $\tilde{f}(t)$. Then c_n and $F(j\omega)$ are related as follows:

$$F(j\omega) = Tc_n\big|_{\substack{n\omega_0 \to \omega \\ T \to \infty}} \tag{8-41}$$

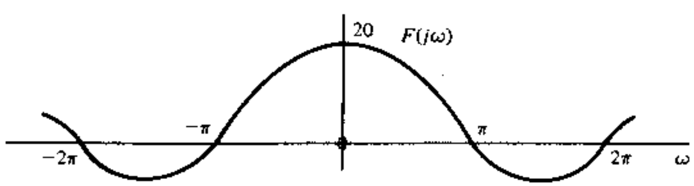


Figure 8-12 Plot of $F(j\omega)$ versus ω for Example 8-11.

and
$$c_n = \frac{1}{T}F(j\omega)\big|_{\omega \to n\omega_0} \tag{8-42}$$

This procedure can make the determination of the Fourier transform a trivial matter. However, it presupposes the existence of the corresponding complex exponential Fourier series coefficients. If these are not available, then we must revert to the defining equation, Equation 8-39, or look up the result in a table of transform pairs. We present a table subsequently.

What does this Fourier transform do? Why use it? What does it mean? Like the Fourier series coefficients, the Fourier transform reveals the spectral content of a signal. It will not normally indicate that some $f(t)$ contains specific frequencies, say, at ω_1, ω_2, and ω_3, but rather, it shows a range of frequencies, say, $\omega_1 < \omega < \omega_2$, over which $f(t)$ contains significant spectral content. If within this range $F(j\omega)$ hits a very narrow peak, say, $\omega \approx \omega_x$, this often indicates the presence of a sinusoid of that specific frequency. This sinusoid might be buried in noise to form $f(t)$ as some data record of "signal plus noise." To ferret signals out of given data records, numerous techniques have been developed from what is called spectral estimation theory. Such studies are beyond our current scope. However, the basics of the Fourier transform are essential to this area and to many fields of sophisticated research in engineering and science.

The Fourier transform is generally a complex function of frequency. We can write:

$$F(j\omega) = |F(j\omega)|\angle\arg F(j\omega) \tag{8-43}$$

A plot of the amplitude spectrum is typically all we need to have a good idea of the spectral content of a given signal. But in order to return from the frequency domain to find $f(t)$, we need both magnitude and phase of $F(j\omega)$. To obtain $f(t)$, given an analytical expression for $F(j\omega)$, we do not normally use the inverse Fourier transform equation, Equation 8-40. Usually, as with inverse Laplace transforms, we would try to break up a given $F(j\omega)$ into terms that are readily inverse transformable, for example, by observation of simple terms that might appear in a table of transform pairs.

EXAMPLE 8-12

Determine the Fourier transform of the following functions:

(a) $f(t) = e^t[u(t) - u(t-1)]$
(b) $f(t) = e^{5t}u(-t) + e^{-t}u(t)$
(c) $f(t) = \sqcap(0.5(t-2))$
(d) $f(t) = te^{-t}u(t)$
(e) $f(t) = \delta(t - t_0)$
(f) $f(t) = u(t+1) - 2u(t) + u(t-1)$
(g) $f(t) = tu(t)$

Solution

(a) $F(j\omega) = \int_0^1 e^t e^{-j\omega t}\,dt = \dfrac{1}{1-j\omega}(e^{1-j\omega} - 1)$

Now from Equation 8-42

$$c_n = \frac{1}{T}F(j\omega)\big|_{\omega\to n\omega_0} = \frac{1}{T}\left(\frac{1}{1-jn\omega_0}\right)(e^{1-jn\omega_0} - 1)$$

but $T = 1$ and $\omega_0 = 2\pi/T = 2\pi$ and $e^{-jn\omega_0} = e^{-jn2\pi} = 1$

Therefore $$e^{1-jn\omega_0} = (e)(1) = 2.718$$

Thus $$c_n = \frac{1.718}{1 - jn2\pi}$$

These c_n values are the exponential Fourier series coefficients for the periodic signal $f(t) = e^t, 0 \le t \le 1$ which has a period $T = 1$.

(b) $F(j\omega) = \int_{-\infty}^{0} e^{5t}e^{-j\omega t}\,dt + \int_0^{\infty} e^{-t}e^{-j\omega t}\,dt$

$$= \frac{1}{5-j\omega}(1-0) + \frac{1}{-1-j\omega}(0-1)$$

$$= \frac{1}{5-j\omega} + \frac{1}{1+j\omega} = \frac{6}{(5-j\omega)(1+j\omega)}$$

(c) Recall from Chapter 1 that

$$\sqcap(t) = 1, \quad \text{for } -0.5 \le t \le 0.5$$
$$= 0, \quad \text{otherwise}$$

Therefore $$\sqcap(0.5(t-2)) = 1, \quad \text{for } 1 \le t \le 3$$
$$= 0, \quad \text{otherwise}$$

and $$F(j\omega) = \int_1^3 1e^{-j\omega t}\,dt = \frac{-1}{j\omega}(e^{-3j\omega} - e^{-j\omega})$$

$$= \frac{e^{-2j\omega}(e^{j\omega} - e^{-j\omega})}{2j(\omega)}(2) = \frac{2e^{-2j\omega}}{\omega}\text{Sin}\,\omega$$

(d) $F(j\omega) = \int_0^{\infty} te^{-t}e^{-j\omega t}\,dt = \frac{e^{at}}{a^2}(at-1)\big|_0^{\infty}, \quad a = (-1 - j\omega)$

$$= 0 - \frac{1}{a^2}(-1) = \frac{1}{a^2} = \frac{1}{(1+j\omega)^2}$$

(e) $F(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0)e^{-j\omega t}\,dt = e^{-j\omega t_0}$

(from the properties of the impulse function)

(f) $F(j\omega) = \int_{-1}^{0} e^{-j\omega t}\,dt + \int_0^1 (-1)e^{-j\omega t}\,dt$

$$= \frac{-1}{j\omega}\{1 - e^{j\omega}\} + \frac{1}{j\omega}\{e^{-j\omega} - 1\}$$

$$= \frac{-2}{j\omega} + \frac{1}{j\omega}(e^{j\omega} + e^{-j\omega}) = \frac{1}{j\omega}\{-2 + 2\,\text{Cos}\,\omega\} = \frac{-4\,\text{Sin}^2(\omega/2)}{j\omega}$$

$$= j\omega\left(\frac{\text{Sin}\,(\omega/2)}{\omega/2}\right)^2$$

(g) $F(j\omega) = \int_0^{\infty} te^{-j\omega t}\,dt = \frac{e^{at}}{a^2}(at-1)\big|_0^{\infty}, \quad a = -j\omega$

$$= \frac{e^{-j\omega\infty}(-j\omega\infty - 1)}{-\omega^2} - \frac{(-1)}{-\omega^2}$$

which is undefined. Thus the Fourier transform of $tu(t)$ does not exist.

The last part of Example 8-12 illustrates that the existence of the Fourier transform, like the Fourier series, is contingent on certain conditions. In order for $f(t)$ to have a Fourier transform, it is sufficient that $f(t)$ have a finite number of maxima, minima, and finite discontinuities in any finite interval. Most functions of interest to engineers will satisfy these restrictions. Another sufficient condition—which is often problematic—is that $f(t)$ be absolutely integrable:

$$\int_{-\infty}^{\infty} |f(t)|\,dt < \infty \qquad (8\text{-}44)$$

These sufficiency conditions for the existence of the Fourier transform, as was the case with the Fourier series, are called Dirichlet conditions. A typical signal encountered in engineering work, like a burst signal from a radar, will satisfy the integrability condition because such a signal starts and stops at finite time points and always has a finite value. However, some simple functions, like $u(t)$, do not satisfy the condition of absolute integrability. Still, by indirect procedures and assuming the existence of $\delta(\omega)$ in the frequency domain, Fourier transforms for such functions can be developed. Functions like the unit step are called *power signals* and are distinguished from *energy signals*.

Energy Signals These are functions $f(t)$ such that $\int_{-\infty}^{\infty} f^2(t)\,dt < \infty$. The integral of a function squared is often taken as a measure of the energy contained in the signal. **Energy signals,** then, are functions that represent finite energy phenomena.

Power Signals These are functions $f(t)$ such that $\lim_{\tau\to\infty} 1/\tau \int_{-\tau/2}^{\tau/2} f^2(t)\,dt < \infty$. Typical examples of these are periodic signals, dc wave forms, and the unit step function. **Power signals** will have infinite energy but will have finite power, whereas energy signals will have finite energy but will have zero power.

Now, in general, an energy signal will die out as $t \to \pm\infty$. The functions considered in Example 8-12 were all energy signals, except for the last function. Signals with finite energy also satisfy the Dirichlet condition of absolute integrability. Their Fourier transforms can be directly computed. The signal $f(t) = tu(t)$ from Example 8-12(g) is neither an energy signal nor a power signal.

Compute $\lim_{\tau\to\infty} 1/\tau \int_0^{\tau/2} t^2\,dt = \lim_{\tau\to\infty} 1/\tau\, \frac{1}{3}(\tau^3/8) = \infty$ which is not finite. If we permit frequency-domain impulse functions, then any signal that is either a power or an energy signal will have a Fourier transform and any signal that is neither an energy nor a power signal will not have a Fourier transform. The unit step function is not an energy signal but it is a power signal and does have a Fourier transform.

EXAMPLE 8-13

Show that $u(t)$ is a power signal and determine its Fourier transform.

Solution. Compute

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^{\tau/2} (1)\,dt = \lim_{\tau\to\infty} \frac{1}{\tau}\left(\frac{\tau}{2}\right) = \frac{1}{2}$$

which is finite. Therefore $u(t)$ is a power signal. Now if we decompose $u(t)$ into its even and odd components, we can write:

$u(t) = \frac{1}{2}f_1(t) + \frac{1}{2}f_2(t)$, where $f_1(t) = 1$ for $-\infty < t < \infty$ is even

and

$$f_2(t) = -1, \quad \text{for } t < 0$$
$$= 0, \quad \text{for } t = 0$$
$$= 1, \quad \text{for } t > 0$$

which is an odd function. This $f_2(t)$ function is sometimes called the **signum function:** $f_2(t) = \text{sgn}(t)$. Taking the Fourier transform, we obtain:

$$\text{FT}\{u(t)\} = \tfrac{1}{2}[\text{FT}\{f_1(t)\} + \text{FT}\{f_2(t)\}]$$

To get the Fourier transforms of $f_1(t)$ and $f_2(t)$, we represent these time functions as limiting processes:

$$f_1(t) = \lim_{a\to 0} e^{at}, \quad t \le 0$$
$$= \lim_{a\to 0} e^{-at}, \quad t \ge 0$$

and

$$f_2(t) = \lim_{a\to 0} -e^{at}, \quad t < 0$$
$$= 0, \quad t = 0$$
$$= \lim_{a\to 0} e^{-at}, \quad t > 0$$

Then

$$F_1(j\omega) = \int_{-\infty}^{\infty} f_1(t)e^{-j\omega t}\,dt$$
$$= \lim_{a\to 0} \int_{-\infty}^{0} e^{at} e^{-j\omega t}\,dt + \lim_{a\to 0} \int_0^{\infty} e^{-at} e^{-j\omega t}\,dt$$
$$= \lim_{a\to 0}\left[\frac{1}{a-j\omega}(e^0 - e^{-\infty}) + \frac{1}{-a-j\omega}(e^{-\infty} - e^0)\right]$$
$$= \lim_{a\to 0}\left[\frac{1}{a-j\omega} + \frac{1}{a+j\omega}\right] = \lim_{a\to 0}\left[\frac{2a}{a^2+\omega^2}\right]$$

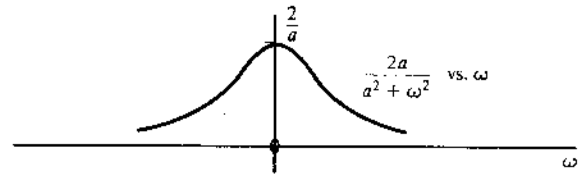


Figure 8-13 Plot of $2a/(a^2 + \omega^2)$ versus ω for Example 8-13.

For a positive finite value of a, if we plot the term in brackets versus ω, we get a function like the one in Figure 8-13. The area under this curve, from a table of definite integrals, is 2π, independent of the value of a. As a gets smaller and smaller, since the peak is $2/a$ at $\omega = 0$, the curve gets sharper and sharper with $2/a \to \infty$ as $a \to 0$. Since the area remains fixed, we end up with an impulse of weight 2π centered at the origin in the ω domain.

$$F_1(j\omega) = 2\pi\delta(\omega)$$

In words, the Fourier transform of a constant is an impulse in the frequency domain.

Now

$$F_2(j\omega) = \int_{-\infty}^{\infty} f_2(t)e^{-j\omega t}\,dt$$
$$= \lim_{a\to 0}\int_{-\infty}^{0} -e^{at}e^{-j\omega t}\,dt + \lim_{a\to 0}\int_0^{\infty} e^{-at}e^{-j\omega t}\,dt$$
$$= \lim_{a\to 0}\left[-\frac{1}{a-j\omega}(e^0 - e^{-\infty}) + \frac{1}{-a-j\omega}(e^{-\infty} - e^0)\right]$$
$$= \lim_{a\to 0}\left[\frac{-1}{a-j\omega} + \frac{1}{a+j\omega}\right]$$
$$= \lim_{a\to 0}\frac{-2j\omega}{a^2+\omega^2} = \frac{-2j}{\omega} = \frac{2}{j\omega}$$

The Fourier transform of the signum function is $2/j\omega$.

Therefore

$$\text{FT}\{u(t)\} = \frac{1}{2}\left\{2\pi\delta(\omega) + \frac{2}{j\omega}\right\}$$

or

$$u(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

The Fourier transform exists for many other power signals. Some of these are easily determined by employing some of the properties of the Fourier transform. Properties of the Fourier transform are the topic of the next section. Before turning to that material, note the summary of Fourier transform pairs presented in Table 8-1. Many of these could be worked out as additional exercises. We will do number 17 as a final example in this section.

TABLE 8-1 FOURIER TRANSFORM TABLE

$f(t)$	$\leftrightarrow$	$F(j\omega)$
1. $\delta(t)$		1
2. 1		$2\pi\delta(\omega)$
3. $u(t)$		$\pi\delta(\omega) + \frac{1}{j\omega}$
4. $e^{-\alpha t}u(t)$		$1/(j\omega + \alpha), \quad \alpha > 0$
5. $t^n e^{-\alpha t}u(t)$		$n!/(\alpha + j\omega)^{n+1}, \quad \alpha > 0$
6. $\lvert t \rvert$		$\frac{-2}{\omega^2}$
7. $\text{Sin}\, \omega_0 t$		$j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$
8. $\text{Cos}\, \omega_0 t$		$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
9. $\frac{\text{Sin}\, \omega_0 t}{\pi t}$		$\begin{cases} 1, & \lvert\omega\rvert < \omega_0 \\ 0, & \lvert\omega\rvert > \omega_0 \end{cases}$
10. $\begin{cases} 1, & \lvert t\rvert < T \\ 0, & \lvert t\rvert > T \end{cases}$		$\frac{2\,\text{Sin}\, \omega T}{\omega}$
11. $e^{j\omega_0 t}$		$2\pi\delta(\omega - \omega_0)$
12. $\delta(t - t_0)$		$e^{-j\omega t_0}$
13. $e^{-\alpha t}\,\text{Cos}\, \omega_0 t u(t)$		$\frac{\alpha + j\omega}{(\alpha + j\omega)^2 + \omega_0^2}$
14. $e^{-\alpha t}\,\text{Sin}\, \omega_0 t u(t)$		$\frac{\omega_0}{(\alpha + j\omega)^2 + \omega_0^2}$
15. $e^{-\alpha^2 t^2}$		$\frac{\sqrt{\pi}}{\alpha} e^{-\omega^2/4\alpha^2}$
16. $e^{-\alpha\lvert t\rvert}, \quad \alpha > 0$		$\frac{2\alpha}{\alpha^2 + \omega^2}$
17. $\text{Cos}\, \omega_0 t[u(t + T) - u(t - T)]$		$\left[\frac{\text{Sin}\,(\omega - \omega_0)T}{(\omega - \omega_0)} + \frac{\text{Sin}\,(\omega + \omega_0)T}{(\omega + \omega_0)}\right]$
18. $\begin{cases} A\left[1 - \frac{\lvert t\rvert}{T}\right], & \lvert t\rvert < T \\ 0, & \lvert t\rvert > T \end{cases}$		$AT\left[\frac{\text{Sin}\, \omega T/2}{\omega T/2}\right]^2$

EXAMPLE 8-14

Determine the Fourier transform for:

$$f(t) = \text{Cos}\, \omega_0 t[u(t + T) - u(t - T)]$$

Solution

$$F(j\omega) = \int_{-\infty}^{\infty} \left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right) e^{-j\omega t}\{u(t + T) - u(t - T)\}\, dt$$

$$= \frac{1}{2}\int_{-T}^{T} (e^{jt(\omega_0 - \omega)} + e^{-jt(\omega_0 + \omega)})\, dt$$

$$= \frac{1}{2}\left\{\frac{1}{j(\omega_0 - \omega)}[e^{jT(\omega_0 - \omega)} - e^{-jT(\omega_0 - \omega)}] + \frac{1}{-j(\omega_0 + \omega)}[e^{-jT(\omega_0 + \omega)} - e^{jT(\omega_0 + \omega)}]\right\}$$

This can be written as:

$$F(j\omega) = \frac{\text{Sin}(\omega - \omega_0)T}{\omega - \omega_0} + \frac{\text{Sin}(\omega_0 + \omega)T}{\omega_0 + \omega}$$

which is the result presented in Table 8-1. However, note that:

$$\omega_0 = \frac{2\pi}{T}$$

Therefore

$$\omega_0 T = 2\pi \quad \text{and} \quad e^{j\omega_0 T} = e^{-j\omega_0 T} = 1$$

Thus

$$F(j\omega) = \frac{1}{2}\left\{\frac{1}{j(\omega_0 - \omega)}[e^{-j\omega T} - e^{j\omega T}] - \frac{1}{j(\omega_0 + \omega)}[e^{-j\omega T} - e^{j\omega T}]\right\}$$

$$= \left(\frac{e^{-j\omega T} - e^{j\omega T}}{2j}\right)\left(\frac{1}{\omega_0 - \omega} - \frac{1}{\omega_0 + \omega}\right)$$

$$= -\text{Sin}\, \omega T\left(\frac{\omega_0 + \omega - \omega_0 + \omega}{\omega_0^2 - \omega^2}\right)$$

or

$$F(j\omega) = \frac{2\omega \sin \omega T}{\omega^2 - \omega_0^2}$$

which is a simplified version.

8-4 FOURIER TRANSFORM PROPERTIES

In Table 8-2 we list some of the more important Fourier transform properties. These properties are labor-saving devices that enable us to determine Fourier transforms or inverse Fourier transforms with a minimum of effort. Employing these properties not only saves work but often provides significant insights into complicated problems. We now prove and demonstrate the use of a number of these properties.

EXAMPLE 8-15

Prove the evenness and oddness property.

Solution. First assume $f(t)$ is even.

Then $F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} f(t)(\text{Cos}\,\omega t - j\,\text{Sin}\,\omega t)$

Now $f(t)$ Cos ωt is an even function and $f(t)$ Sin ωt is odd. The integrating from $-\infty$ to $+\infty$, we get:

$$F(j\omega) = 2\int_0^{\infty} f(t)\,\text{Cos}\,\omega t\,dt + 0$$

and $F(j\omega)$ is even.
Now assume $f(t)$ is odd.

Then $$F(j\omega) = \int_{-\infty}^{\infty} f(t)(\text{Cos}\,\omega t - j\,\text{Sin}\,\omega t)\,dt$$

$$= 0 - j2\int_0^{\infty} f(t)\,\text{Sin}\,\omega t\,dt$$

and $F(j\omega)$ is odd.

EXAMPLE 8-16.
Prove the time shift property, then use it to determine the Fo transform of:

$$f(t) = \sqcap(t) = u(t + \tfrac{1}{2}) - u(t - \tfrac{1}{2})$$

Solution. We know:

$$\text{FT}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

Then $$\text{FT}\{f(t - t_0)\} = \int_{-\infty}^{\infty} f(t - t_0)e^{-j\omega t}\,dt$$

Let $$t - t_0 = \lambda, \qquad dt = d\lambda$$

and $$\int_{-\infty}^{\infty} f(t - t_0)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} f(\lambda)e^{-j\omega(\lambda + t_0)}\,d\lambda$$

$$= e^{-j\omega t_0}\int_{-\infty}^{\infty} f(\lambda)e^{-j\omega\lambda}\,d\lambda$$

Therefore $$e^{-j\omega t_0}F(j\omega) \leftrightarrow f(t - t_0)$$

Now we know:

$$u(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

Thus $$u\left(t + \frac{1}{2}\right) \leftrightarrow e^{j\omega 1/2}\left\{\pi\delta(\omega) + \frac{1}{j\omega}\right\},$$

$$u\left(t - \frac{1}{2}\right) \leftrightarrow e^{-j\omega 1/2}\left\{\pi\delta(\omega) + \frac{1}{j\omega}\right\}$$

and $$\sqcap(t) \leftrightarrow \left(e^{j\omega 1/2}\pi\delta(\omega) + \frac{e^{j\omega 1/2}}{j\omega}\right)$$

$$- \left(e^{-j\omega 1/2}\pi\delta(\omega) + \frac{e^{-j\omega 1/2}}{j\omega}\right)$$

TABLE 8-2 FOURIER TRANSFORM PROPERTIES

Given $f(t) \leftrightarrow F(j\omega)$, $g(t) \leftrightarrow G(j\omega)$

Function	Transform	Property
1. $\alpha f(t) + \beta g(t)$	$\alpha F(j\omega) + \beta G(j\omega)$	Linearity
2. $f(t)$ even	$F(j\omega) = 2\int_0^{\infty} f(t)\,\text{Cos}\,\omega t\,dt$	Evenness and
$f(t)$ odd	$F(j\omega) = -j2\int_0^{\infty} f(t)\,\text{Sin}\,\omega t\,dt$	Oddness
3. $f(t - t_0)$	$e^{-j\omega t_0}F(j\omega)$	Time shift
4. $f(\alpha t)$	$\frac{1}{\lvert\alpha\rvert}F\left(j\frac{\omega}{\alpha}\right)$	Time scale
5. $F(jt)$	$2\pi f(-\omega)$	Duality
6. $f(t) * g(t)$	$F(j\omega)G(j\omega)$	Time convolution
7. $f(t)g(t)$	$\frac{1}{2\pi}F(j\omega) * G(j\omega)$	Frequency convolution
8. $\frac{d^n}{dt^n}f(t)$	$(j\omega)^n F(j\omega)$	Time differentiation
9. $\int_{-\infty}^{t} f(\lambda)\,d\lambda$	$\frac{1}{j\omega}F(j\omega) + \pi F(0)\delta(\omega)$	Integration
10. $t^n f(t)$	$\frac{(j)^n d^n}{d\omega^n}F(j\omega)$	Frequency differentiation
11. $e^{j\omega_0 t}f(t)$	$F(j[\omega - \omega_0])$	Modulation (frequency shift)
12. $\int_{-\infty}^{\infty} f(\lambda - t)g(\lambda)\,d\lambda$	$F(-j\omega)G(j\omega)$	Correlation

but $$e^{j\omega 1/2}\pi\delta(\omega) = e^{-j\omega 1/2}\pi\delta(\omega) = \pi\delta(\omega)$$

Therefore $$\text{FT}\{\sqcap(t)\} = \frac{1}{j\omega}(e^{j\omega 1/2} - e^{-j\omega 1/2})$$

$$= \frac{2}{\omega}\,\text{Sin}\,\omega\frac{1}{2}$$

which agrees with the result of number 10 from Table 8-1 with $T = \frac{1}{2}$.

EXAMPLE 8-17
The time scale property, which is sometimes known as the reciprocal spreading property, indicates that an expansion in the time domain results in a contraction in the frequency domain and vice versa. Demonstrate this result by determining and plotting $f(10t)$ and $f(\frac{1}{2}t)$ where $f(t)$ is the triangular function shown in Figure 8-14.

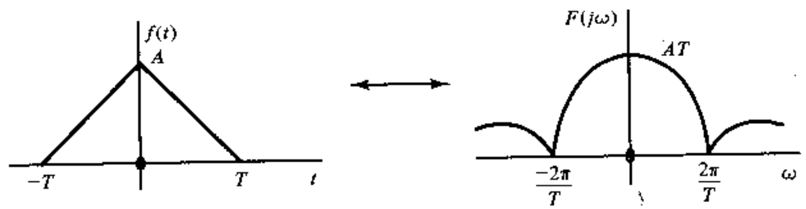


Figure 8-14 Fourier transform pair for Example 8-17.

Solution. Plots of $f(10t)$ and $f(\frac{1}{2}t)$ appear in Figure 8-15. From Property 4, $f(10t) \leftrightarrow \frac{1}{10}F(j(\omega/10)$ and $f(\frac{1}{2}t) \leftrightarrow 2F(j2\omega)$. Plots of $\frac{1}{10}F(j\omega/10$ and $2F(j2\omega)$ appear in Figure 8-16. Note that if the time function is contracted, then the transform will be expanded. If the time function is expanded, then its transform will be contracted.

EXAMPLE 8-18

(a) Prove the duality property.
(b) Use it to determine the Fourier transform of $f(t) = 10/(t^2 + 1)$.
(c) Use it to determine the Fourier transform of $f(t) = \text{Sin}\, t/t$.

Solution

(a) $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\, d\omega$

Changing the dummy variable ω to x, we obtain:

$$2\pi f(t) = \int_{-\infty}^{\infty} F(jx)e^{jxt}\, dx$$

Now replace t by $-\omega$ to yield:

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(jx)e^{-jx\omega}\, dx$$

Now on the right-hand side change the dummy variable x to t which gives:

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(jt)e^{-j\omega t}\, dt = \text{FT}\{F(jt)\}$$

Therefore $\qquad F(jt) \leftrightarrow 2\pi f(-\omega)$

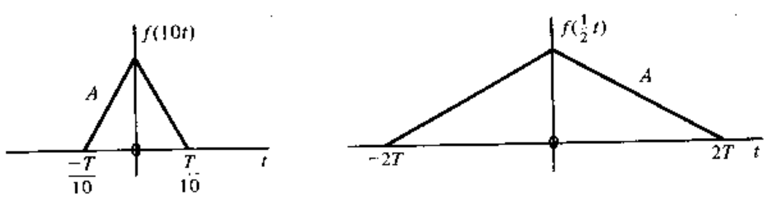


Figure 8-15 Plots of $f(10t)$ and $f(\frac{1}{2}t)$ for Example 8-17.

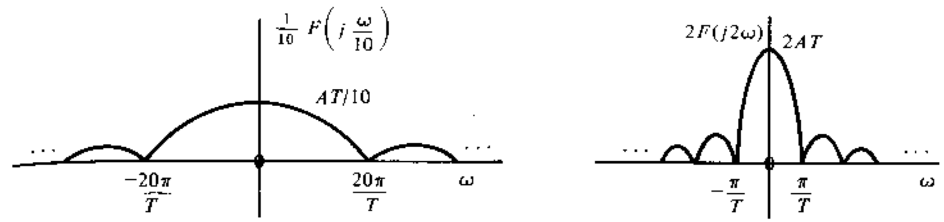


Figure 8-16 Plots of $\frac{1}{10}F(j\frac{\omega}{10})$ and $2F(j2\omega)$ for Example 8-17.

(b) We know that:

$$e^{-|t|} \leftrightarrow \frac{2}{\omega^2 + 1}$$

where $\qquad f(t) = e^{-|t|}$ and $F(j\omega) = \frac{2}{\omega^2 + 1}$

therefore $\qquad F(jt) = \frac{2}{t^2 + 1} \leftrightarrow 2\pi f(-\omega) = 2\pi e^{-|-\omega|} = 2\pi e^{-|\omega|}$

Thus $\qquad \frac{10}{t^2 + 1} \leftrightarrow 10\pi e^{-|\omega|}$

(c) We know that:

$$f(t) = u(t + T) - u(t - T) \leftrightarrow F(j\omega) = \frac{2\, \text{Sin}\, \omega T}{\omega}$$

Therefore $\qquad F(jt) = \frac{2\, \text{Sin}\, tT}{t} \leftrightarrow 2\pi f(-\omega)$

Since $\qquad f(-t) = f(t), \quad f(-\omega) = u(\omega + T) - u(\omega - T)$

If $T = 1$:

$$\frac{2\, \text{Sin}\, t}{t} \leftrightarrow 2\pi[u(\omega + 1) - u(\omega - 1)]$$

Thus $\qquad \frac{\text{Sin}\, t}{t} \leftrightarrow \pi[u(\omega + 1) - u(\omega - 1)]$

Note that this result checks with number 9 in Table 8-1.

As mentioned earlier, we use $F(j\omega)$ instead of $F(\omega)$ or $F(f)$ for the Fourier transform. However, the $F(f)$ notation provides an interesting symmetry when employed in the duality property. Let:

$$F(f) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}\, dt \qquad (8\text{-}45)$$

which is just $F(j\omega)$ with $\omega = 2\pi f$.

$$f(t) = \int_{-\infty}^{\infty} F(f)e^{j2\pi ft}\, df \tag{8-46}$$

Note the absence of the 2π term in the inverse Fourier transform. Using the $F(f)$ notation, we find that the 2π term is also absent in the statement of the duality property: If $f(t) \leftrightarrow F(f)$, then $F(t) \leftrightarrow f(-f)$. From the previous example, for instance:

$$e^{-|t|} \leftrightarrow \frac{2}{\omega^2+1} \quad \text{and} \quad \frac{1}{t^2+1} \leftrightarrow \pi e^{-|\omega|};$$

But $F(f) = F(j\omega)|_{\omega=2\pi f}$.

Thus $$\frac{2}{(2\pi f)^2+1} \leftrightarrow e^{-|t|} \quad \text{and} \quad \frac{2}{(2\pi t)^2+1} \leftrightarrow e^{-|f|}$$

Both versions of the duality property then will yield similar results. The differences lie primarily in scaling.

EXAMPLE 8-19

Prove the time convolution property and use it to determine the system input when the system has an impulse response

$$h(t) = e^{-10t}u(t)$$

and the system output is:

$$y(t) = (e^{-5t} - e^{-15t})u(t)$$

Solution

$$\text{FT}\{f(t)*g(t)\} = \int_{-\infty}^{\infty} f(t)*g(t)e^{-j\omega t}\, dt$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\lambda)g(t-\lambda)e^{-j\omega t}\, dt\, d\lambda$$

Let $t - \lambda = v$, then $dt = dv$

and $$f(t)*g(t) \leftrightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\lambda)g(v)e^{-j\omega v}e^{-j\omega\lambda}\, d\lambda\, dv$$
$$= \int_{-\infty}^{\infty} f(\lambda)e^{-j\omega\lambda}d\lambda \int_{-\infty}^{\infty} g(v)e^{-j\omega v}\, dv$$
$$= F(j\omega)G(j\omega)$$

Therefore $$f(t)*g(t) \leftrightarrow F(j\omega)G(j\omega)$$

Now we know that

$$y(t) = h(t)*x(t)$$

The time convolution property indicates that the Fourier transform of the output is:

$$Y(j\omega) = H(j\omega)X(j\omega)$$

The system function:

$$H(j\omega) = \frac{1}{j\omega+10} \quad \text{and} \quad Y(j\omega) = \frac{1}{j\omega+5} - \frac{1}{j\omega+15}$$
$$Y(j\omega) = \frac{10}{(j\omega+5)(j\omega+15)}$$

Then $$X(j\omega) = \frac{10/(j\omega+5)(j\omega+15)}{1/(j\omega+10)} = \frac{10(j\omega+10)}{(j\omega+5)(j\omega+15)}$$

Next, using partial fraction expansion, we can write:

$$X(j\omega) = \frac{A}{j\omega+5} + \frac{B}{j\omega+15} = \frac{5}{j\omega+5} + \frac{5}{j\omega+15}$$

Therefore $$x(t) = (5e^{-5t} + 5e^{-15t})u(t)$$

EXAMPLE 8-20

Use the frequency convolution property to verify number 13 in Table 8-1.

Solution

$$f_1(t) = e^{-\alpha t}\,\text{Cos}\,\omega_0 t\, u(t)$$

Let:

$$f(t) = e^{-\alpha t}u(t) \quad \text{and} \quad g(t) = \text{Cos}\,\omega_0 t$$
$$F(j\omega) \leftrightarrow \frac{1}{j\omega+\alpha} \quad \text{and} \quad G(j\omega) \leftrightarrow \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

Then $$F_1(j\omega) = \frac{1}{2\pi}F(j\omega)*G(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\lambda)G(j[\omega-\lambda])\, d\lambda$$
$$= \frac{\pi}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{j\lambda+\alpha}\right)(\delta(\omega-\lambda-\omega_0) + \delta(\omega-\lambda+\omega_0))\, d\lambda$$
$$= \frac{1}{2}\left\{\frac{1}{j(\omega-\omega_0)+\alpha} + \frac{1}{j(\omega+\omega_0)+\alpha}\right\}$$
$$= \frac{j\omega+\alpha}{[j(\omega-\omega_0)+\alpha][j(\omega+\omega_0)+\alpha]}$$
$$F_1(j\omega) = \frac{j\omega+\alpha}{(\alpha+j\omega)^2+\omega_0^2}$$

EXAMPLE 8-21

Prove the time differentiation property and use it to determine the Fourier transform of the $f(t)$ in Figure 8-17.

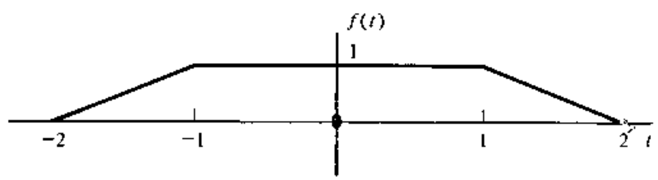


Figure 8-17 Time function used in Example 8-21.

Solution. The inverse Fourier transform is:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega$$

Then $\dfrac{df(t)}{dt} = \dfrac{d}{dt}\left\{\dfrac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega\right\} = \int_{-\infty}^{\infty}\dfrac{1}{2\pi}j\omega F(j\omega)e^{j\omega t}\,d\omega$

Therefore $\dot{f}(t) \leftrightarrow j\omega F(j\omega)$

Likewise, $\ddot{f}(t) \leftrightarrow (j\omega)^2 F(j\omega)$

and in general:

$$\frac{d^n f(t)}{dt^n} \leftrightarrow (j\omega)^n F(j\omega)$$

Now if we differentiate $f(t)$, then differentiate again, we obtain the plot indicated in Figure 8-18. The second derivative consists of four impulses. Impulses have very simple transforms:

$\delta(t - t_0) \leftrightarrow e^{-j\omega t_0}$

Therefore:

$$\ddot{f}(t) \leftrightarrow e^{+2j\omega} - e^{j\omega} - e^{-j\omega} + e^{-2j\omega} = 2\,\text{Cos}\,2\omega - 2\,\text{Cos}\,\omega$$

But this is $(j\omega)^2 F(j\omega)$.

Thus $F(j\omega) = \dfrac{2\,\text{Cos}\,2\omega - 2\,\text{Cos}\,\omega}{(j\omega)^2}$

or $F(j\omega) = \dfrac{2\,\text{Cos}\,\omega - 2\,\text{Cos}\,2\omega}{\omega^2}$

This procedure is often useful: (1) Given $f(t)$, (2) differentiate $f(t)$ enough

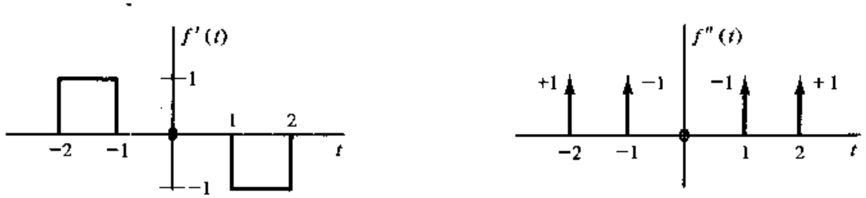


Figure 8-18 First and second derivatives of $f(t)$.

times to yield only impulses or their derivatives, (3) transform, and (4) divide by $(j\omega)^k$ where k is the number of derivatives performed.

EXAMPLE 8-22

Use the frequency differentiation property to determine the Fourier transform of the following:

(a) $f_1(t) = te^{-5t}u(t)$
(b) $f_2(t) = te^{-t^2}$
(c) $f_3(t) = te^{-|t|}$
(d) $f_4(t) = t^2e^{t}u(-t)$
(e) $f_5(t) = tu(t)$

Solution

(a) Let:

$$tf(t) = te^{-5t}u(t).$$

$$f(t) = e^{-5t}u(t) \leftrightarrow \frac{1}{5 + j\omega}$$

Then $tf(t) \leftrightarrow j\dfrac{d}{d\omega}\left(\dfrac{1}{5 + j\omega}\right) = j(-1)(5 + j\omega)^{-2}(j)$

Therefore $te^{-5t}u(t) \leftrightarrow \dfrac{1}{(j\omega + 5)^2}$

(b) Let:

$$te^{-t^2} = tf(t) \leftrightarrow j\frac{d}{d\omega}F(j\omega)$$

where $F(j\omega) = \sqrt{\pi}e^{-\omega^2/4}$

$$\frac{d}{dx}e^u = e^u\frac{du}{dx} \to \frac{d}{d\omega}e^{-\omega^2/4}$$

$$= e^{-\omega^2/4}\frac{d}{d\omega}\left(-\frac{1}{4}\omega^2\right)$$

$$= e^{-\omega^2/4}\left(-\frac{1}{2}\omega\right)$$

Thus $te^{-t^2} \leftrightarrow -j\dfrac{\omega}{2}\sqrt{\pi}e^{-\omega^2/4}$

(c) $e^{-|t|} \leftrightarrow \dfrac{2}{1 + \omega^2}$

and $\dfrac{d}{d\omega}2(1 + \omega^2)^{-1} = -2(1 + \omega^2)^{-2}2\omega$

Therefore $$te^{-|t|} \leftrightarrow \frac{-4j\omega}{(1+\omega^2)^2}$$

(d) $$\text{FT}\{e^t u(-t)\} = \int_{-\infty}^{0} e^t e^{-j\omega t}\,dt$$
$$= \frac{1}{1-j\omega}(1-0) = \frac{1}{1-j\omega}$$

Then $$\frac{d}{d\omega}(1-j\omega)^{-1} = -(1-j\omega)^{-2}(-j) = j(1-j\omega)^{-2}$$
$$\frac{d^2}{d\omega^2}(1-j\omega)^{-1} = -2j(1-j\omega)^{-3}(-j) = \frac{-2}{(1-j\omega)^3}$$

and $$t^2 f(t) \leftrightarrow (j)^2 \frac{d^2}{d\omega^2} F(j\omega) = \frac{2}{(1-j\omega)^3}$$

(e) Let:

$$f(t) = u(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega} = F(j\omega)$$

$$\frac{d}{d\omega} F(j\omega) = \pi\dot{\delta}(\omega) - \frac{1}{j\omega^2}$$

Then $$tu(t) \leftrightarrow j\frac{d}{d\omega} F(j\omega) = j\pi\dot{\delta}(\omega) - \frac{1}{\omega^2}$$

Note that Example 8-22(e) presents a dilemma. Part (g) in Example 8-1[illegible] asked for the Fourier transform directly. The conclusion there was that F[illegible] $\{tu(t)\}$ does not exist. The reason was that $tu(t)$ was neither an energy signal whose Fourier transforms are not problematic, nor a power signal, whose Fourier transforms are not problematic as long as we allow $\delta(\omega)$ functions to exist in the frequency domain. Note the result for Example 8-22(e). $F(j\omega)$ contains a unit-doublet. From generalized function theory, which is beyond the scope of this book, it can be shown that if $\dot{f}(t)$ is a power signal *and* we let $\dot{\delta}(\omega)$ exist in the frequency domain, then $F(j\omega)$ can be developed. Likewise, if $\ddot{f}(t)$ is a power signal *and* we let $\ddot{\delta}(\omega)$ exist, then $F(j\omega)$ can be developed, and so on. Due to the abstract nature of these issues, we will not consider them further. We turn instead to an examination of the frequency shift or modulation property. This property proves to be very useful in a number of different areas of communication theory.

EXAMPLE 8-23

Use the modulation property to determine the Fourier transform of the following functions:

(a) $f_1(t) = f(t)\,\text{Cos}\,\omega_c t$

(b) $f_2(t) = f(t)\,\text{Sin}\,\omega_c t$

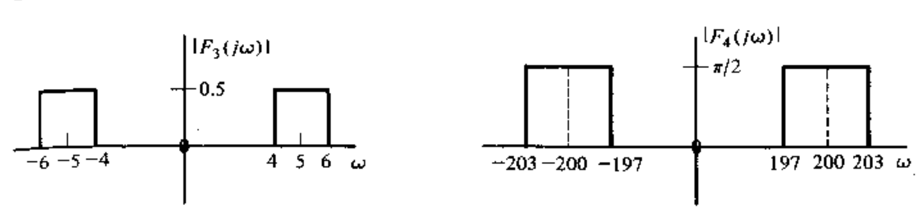


Figure 8-19 Plot of magnitudes of F_3 and F_4 from Example 8-23.

(c) $f_3(t) = \dfrac{\text{Sin}\,t\,\text{Sin}\,5t}{\pi t}$

(d) $f_4(t) = \dfrac{\text{Sin}\,3t\,\text{Cos}\,200t}{t}$

(e) plot $|F_3(j\omega)|$ and $|F_4(j\omega)|$

Solution

(a) $$f_1(t) = f(t)\,\text{Cos}\,\omega_c t = \frac{f(t)}{2}\{e^{j\omega_c t} + e^{-j\omega_c t}\}$$
$$\leftrightarrow \frac{F(j[\omega-\omega_c]) + F(j[\omega+\omega_c])}{2}$$

(b) $$f_2(t) = f(t)\,\text{Sin}\,\omega_c t = \frac{f(t)}{2j}\{e^{j\omega_c t} - e^{-j\omega_c t}\}$$
$$\leftrightarrow \frac{F(j[\omega-\omega_c]) - F(j[\omega+\omega_c])}{2j}$$

(c) $$f_3(t) = \frac{\text{Sin}\,t}{\pi t}\frac{e^{5jt} - e^{-5jt}}{2j} \quad \text{but} \quad \frac{\text{Sin}\,t}{\pi t} \leftrightarrow \sqcap\left(\frac{\omega}{2}\right)$$

Therefore $$F_3(j\omega) = \frac{\sqcap((\omega-\omega_c)/2) - \sqcap((\omega+\omega_c)/2)}{2j}$$
$$= \frac{\sqcap((\omega-5)/2) - \sqcap((\omega+5)/2)}{2j}$$

(d) $$f_4(t) = \frac{\text{Sin}\,3t}{t}\left\{\frac{e^{200jt} + e^{-200jt}}{2}\right\} \quad \textbf{but} \quad \frac{\text{Sin}\,3t}{t} \leftrightarrow \pi\sqcap\left(\frac{\omega}{6}\right)$$

Thus $$F_4(j\omega) = \frac{\pi}{2}\sqcap\left(\frac{\omega-200}{6}\right) + \frac{\pi}{2}\sqcap\left(\frac{\omega+200}{6}\right)$$

(e) Plots of $|F_3(j\omega)|$ and $|F_4(j\omega)|$ appear in Figure 8-19.

EXAMPLE 8-24

Use the modulation property to determine the Fourier transform of an arbitrary periodic $f(t)$ which is represented as a complex exponential Fourier series.

Solution. Let:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

Each of the terms in the summation is a constant multiplied by a complex exponential. The Fourier transform of a constant is an impulse; that is:

$$\mathrm{FT}\{c_n\} = 2\pi c_n \delta(\omega)$$

Therefore the Fourier transform of $f(t)$ is a summation of impulses, each of which is shifted in frequency due to the complex exponential terms; that is

$$\mathrm{FT}\{f(t)\} = 2\pi \sum_{n=-\infty}^{\infty} c_n \delta(\omega - n\omega_0)$$

The Fourier transform of a Fourier series consists of a sequence of impulses. Each impulse is weighted by $2\pi c_n$ and all impulses are separated from each other by ω_0. Although the term ω_0 is similar to the period of the transform, the Fourier transform is not a periodic function. Even though the impulses are all separated by the same amount, their weights are all different. The best way to understand the relationship between the Fourier series and the Fourier transform is to imagine that the line spectra in the Fourier series are replaced by infinite lines or impulses in the Fourier transform. Each Fourier transform impulse is weighted with the corresponding complex exponential Fourier series coefficient c_n (times 2π).

EXAMPLE 8-25

Demonstrate the correlation property.

Solution. The property states that the Fourier transform of;

$$\int_{-\infty}^{\infty} f(\lambda - t) g(\lambda)\, d\lambda \quad \text{is the product} \quad F(-j\omega)G(j\omega)$$

This of course is very similar to the convolution property. The integral expression is written $f(t) \oplus g(t)$ analogous to the convolution notation. In the integral, if we let $\lambda - t = p$, then the integral becomes:

$$\int_{-\infty}^{\infty} f(p) g(\lambda)\, d\lambda$$

The Fourier transform of this integral then is:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(p) g(\lambda) e^{-j\omega t}\, dt\, d\lambda$$

but $dt = -dp$, so we obtain:

$$\int_{-\infty}^{\infty}\int_{+\infty}^{-\infty} f(p) g(\lambda) e^{-j\omega(\lambda - p)}(-dp)\, d\lambda$$

The minus sign with dp reverses the limits on the second integral and we can write:

$$\int_{-\infty}^{\infty} f(p) e^{j\omega p}\left[\int_{-\infty}^{\infty} g(\lambda) e^{-j\omega\lambda}\, d\lambda\right] dp = F(-j\omega)G(j\omega)$$

Now before concluding this section on properties of the Fourier transform, we consider Parseval's theorem. In the Fourier series discussion we discussed what was called Parseval's relation. This equation related the energy contained in a finite time interval of a function to the Fourier series coefficients of that function. Parseval's theorem is similar. Consider a real energy signal $f(t)$ with the Fourier transform $F(j\omega)$. Let the energy contained in $f(t)$ be:

$$\mathscr{E} = \int_{-\infty}^{\infty} f^2(t)\, dt \tag{8-47}$$

Since $f(t) = (1/2\pi)\int_{-\infty}^{\infty} F(j\omega) e^{j\omega t}\, d\omega$ is the inverse Fourier transform of $F(j\omega)$, we can write:

$$\mathscr{E} = \int_{-\infty}^{\infty} f(t)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega) e^{j\omega t}\, d\omega\right\} dt \tag{8-48}$$

which can be further expressed as:

$$\mathscr{E} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\left\{\int_{-\infty}^{\infty} f(t) e^{j\omega t}\, dt\right\} d\omega \tag{8-49}$$

But note that the term in parentheses here is just $F(-j\omega)$, and since we are assuming that $f(t)$ is a real function of time, we know that $F(-j\omega) = F^*(j\omega)$. Also, we know for any complex function that $FF^* = |F|^2$. Thus we have:

$$\mathscr{E} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)F(-j\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2\, d\omega \tag{8-50}$$

Relating time- and frequency-domain integrals, we can write Parseval's theorem:

$$\int_{-\infty}^{\infty} f^2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2\, d\omega \tag{8-51}$$

The term $|F|^2$ is called the **energy spectral density** and indicates a distribution of energy over a spectral band. For instance, if $F(j\omega)$ is fairly constant over a small band $\Delta\omega = \omega_2 - \omega_1$, then the energy contained in that band is approximately $|F|^2\Delta\omega/2\pi$. This result can be obtained from Equation 8-51 if we let F be constant and integrate from ω_1 to ω_2 instead of from $-\infty$ to $+\infty$. Then we have $\mathscr{E} = |F|^2\Delta\omega/2\pi$ as the energy contained in the spectral band $\omega_1 \le \omega \le \omega_2$. We can write $|F|^2 = 2\pi\mathscr{E}/\Delta\omega$. Dividing $\mathscr{E}$ by $\Delta\omega$ gives a kind of energy density: We have an amount of energy per $\Delta\omega$. This is the motivation for calling the term $|F|^2$ the energy spectral density. Note that when we talk about continuous energy spectral densities or continuous Fourier transforms, the energy over a band of frequencies—never the energy contained in a single frequency—is of interest.

With regard to linear systems, the Parseval theorem can be useful. If $f(t) \leftrightarrow F(j\omega)$ is the input, $g(t) \leftrightarrow G(j\omega)$ is the output, and $H(j\omega)$ is the system transfer function, then the output energy spectral density is:

$$|G(j\omega)|^2 = |F(j\omega)|^2|H(j\omega)|^2 \tag{8-52}$$

The term $|H(j\omega)|^2$ is called the **energy transfer function.** It relates the input energy spectral density to the output energy spectral density. Because of the magnitude-squared nature of these terms, the output and the input energy spectral densities are both independent of any phase variations that might be present.

Another use for Parseval's theorem is in what is called **energy localization.** Assume for some given $f(t)$ that the left-hand side of Equation 8-51 can be computed. This yields the total energy contained in the signal. Now on the right-hand side of Equation 8-51, note first that $|F|^2$ is an even function of ω. Thus we can write:

$$\int_{-\infty}^{\infty} f^2(t)\, dt = \frac{1}{\pi}\int_0^{\infty} |F(j\omega)|^2\, d\omega \qquad (8\text{-}53)$$

Often the energy spectrum $|F|^2$ will be concentrated over a finite band of frequencies. A typical question in this area is to determine such a frequency band within which a certain percentage of the total energy will be localized.

EXAMPLE 8-26

Determine a frequency band $(0, \omega_c)$ over which one half the energy in $f(t) = e^{-t}u(t)$ will be localized.

Solution. The energy in $f(t)$ is:

$$\mathcal{E} = \int_{-\infty}^{\infty} f^2(t)\, dt = \int_0^{\infty} e^{-2t}\, dt = 0.5$$

Now, from Equation 8-53, we can write:

$$\frac{1}{2}(0.5) = \frac{1}{\pi}\int_0^{\omega_c} |F|^2\, d\omega$$

equating one half the energy to the integral with finite upper limit. We know that:

$$F(j\omega) = \frac{1}{1 + j\omega}$$

Thus $$|F|^2 = \frac{1}{1 + \omega^2}$$

and $$0.25 = \frac{1}{\pi}\int_0^{\omega_c} \frac{1}{1 + \omega^2}\, d\omega = \frac{1}{\pi}\{\text{Tan}^{-1}\,\omega|_0^{\omega_c}\}$$

or $$0.25\pi = \text{Tan}^{-1}\,\omega_c - \text{Tan}^{-1}\,0 = \text{Tan}^{-1}\,\omega_c$$

therefore $$\omega_c = \text{Tan}\,(\pi/4) = 1 \text{ rad/s}$$

The discussion on Parseval's theorem provides a transition between the properties and the applications of the Fourier transform. The result postulated in Parseval's theorem employs the idea of signal energy and follows directly from the definitions of the Fourier transform and the inverse Fourier transform. Using Parseval's theorem in the energy localization problem introduces Fourier transform applications. Applications of the Fourier transform span a wide variety of disciplines. Some of these applications will be dealt with in Section 8.6.

At this point, we pause in order to consolidate our results. We studied the Fourier series and from it developed the Fourier transform. A number of properties of the Fourier transform were considered, not only as an aid to obtain Fourier transform functions, but also as a means to gain deeper insights into the essence of the Fourier transform. Even further appreciation can be obtained by comparing the Fourier transform to the Laplace transform, which has already been discussed in Chapters 4 and 5. A basic understanding of the Laplace transform is presupposed. The next short section deals with the relationship between the Fourier and Laplace transforms.

8-5 THE FOURIER TRANSFORM AND THE LAPLACE TRANSFORM: A COMPARISON

From a cursory glance at the two transforms we might conclude that $F(j\omega)$ is just $F(s)$ with s replaced by $j\omega$. This, however, is not always the case. It is so if $f(t) = 0,\ t < 0$, and $\int_0^{\infty} |f(t)|\, dt < \infty$; that is, if $f(t)$ is absolutely integrable.

EXAMPLE 8-27

Determine $F(j\omega)$ from $F(s)$ for:

(a) $f_1(t) = e^{-10t}u(t)$
(b) $f_2(t) = e^{-t}\,\text{Cos}\,10t\,u(t)$
(c) $f_3(t) = u(t) - u(t - 10)$

Solution

(a) $$F_1(s) = \frac{1}{s + 10}$$

Since $f_1(t)$ is zero for $t < 0$ and $f_1(t)$ is absolutely integrable:

$$F_1(j\omega) = \frac{1}{10 + j\omega}$$

(b) $$F_2(s) = \frac{s + 1}{(s + 1)^2 + 100}\,, \quad F_2(j\omega) = \frac{j\omega + 1}{(j\omega + 1)^2 + 100}$$

(c) $$F_3(s) = \frac{1}{s} - \frac{1}{s}e^{-10s}\,, \quad F_3(j\omega) = \frac{1}{j\omega} - \frac{1}{j\omega}e^{-10j\omega}$$