Chebyshev polynomials

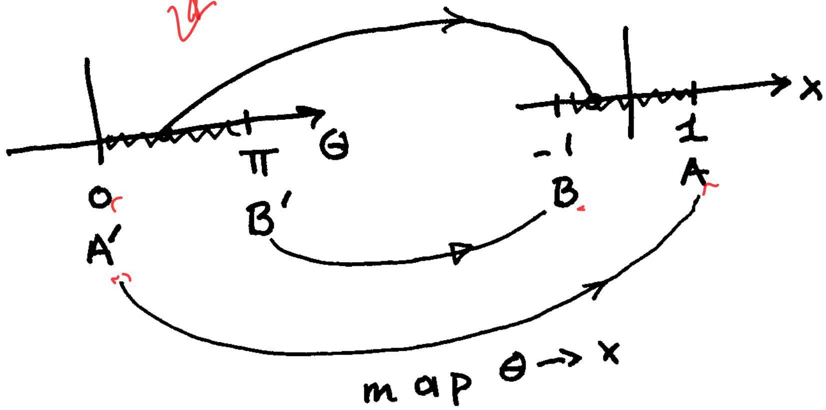
The n^{th} order Chebyshev poly. of the 1^{st} kind is defined

$$T_n(\theta) = \cos(n\theta) \qquad \underbrace{0 \le \theta \le \pi}_{\theta\text{-domain}}$$

If $x = \cos\theta$

$$T_n(x) = \cos(n \cos^{-1} x) \qquad \underbrace{-1 \le x \le 1}_{\text{cheb-domain}}$$

$x = \cos\theta$



where $x = \cos\theta$

$T_0(x) = 1$	$T_0(\theta) = 1$
$T_1(x) = x$	$T_1(\theta) = \cos(\theta)$
$T_2(x) = 2x^2 - 1$	$T_2(\theta) = \cos(2\theta)$
$T_3(x) = 4x^3 - 3x$	$T_3(\theta) = \cos(3\theta)$

Recursion T_n

(a) $\cos(n\theta)\cos\theta = \frac{1}{2}\left[\cos[(n+1)\theta] + \cos[(n-1)\theta]\right]$

$$\boxed{\begin{aligned} &T_n(x)\,x = \frac{1}{2}\left[T_{n+1}(x) + T_{n-1}(x)\right] \qquad n \geq 1 \\ &\text{or} \\ &T_{n+1}(x) = 2T_n(x)\,x - T_{n-1}(x) \end{aligned}}$$

(b)
$$\boxed{\begin{aligned} &\frac{\partial T_{n+1}}{\partial x}\frac{1}{n+1} - \frac{\partial T_{n-1}}{\partial x}\frac{1}{n-1} = 2T_n \qquad n \geq 2 \\ &\frac{\partial T_0}{\partial x} = 0\,;\ \frac{\partial T_1}{\partial x} = 1 \end{aligned}}$$

proof $\frac{\partial T_{n+1}}{\partial x}\frac{1}{n+1} - \frac{\partial T_{n-1}}{\partial x}\frac{1}{n-1} = \frac{1}{\sin\theta}\left[\sin[(n+1)\theta] - \sin[(n-1)\theta]\right]$

note: $2\sin\theta\cos(n\theta) = \sin[(n+1)\theta] - \sin[(n-1)\theta]$

$\frac{\partial T_{n+1}}{\partial x}\frac{1}{n+1} - \frac{\partial T_{n-1}}{\partial x}\frac{1}{n-1} = 2T_n$

Orthogonality

(a) $\int_0^{\pi} T_n(\theta) T_m(\theta)\, d\theta = \int_0^{\pi} \cos(n\theta)\cos(m\theta)\, d\theta$

$$= \frac{1}{2}\int_0^{\pi} \cos((m+n)\theta)\, d\theta + \frac{1}{2}\int_0^{\pi} \cos((m-n)\theta)\, d\theta$$

$$= \begin{cases} 0 & m \neq n \\ \pi/2 & m = n \neq 0 \\ \pi & m = n = 0 \end{cases}$$

In terms of x

$x = \cos\theta$ $\Rightarrow$ $x = \cos\theta$

$dx = -\sin\theta\, d\theta$ $\Rightarrow$ $dx = -\sqrt{1-\cos^2\theta}\, d\theta \Rightarrow d\theta = -\frac{dx}{\sqrt{1-x^2}}$

$$\int_{-1}^{1} \frac{T_n(x) T_m(x)}{\sqrt{1-x^2}}\, dx = \begin{cases} 0 & m \neq n \\ \pi/2 & m = n \neq 0 \\ \pi & m = n = 0 \end{cases}$$

or

$$\int_{-1}^{1} T_n(x) T_m(x) w(x)\, dx$$

where $w(x) = \frac{1}{\sqrt{1-x^2}}$

Discrete Orthognality

A. Extrema sampling (based of maxima of $T_N(\theta)$

$$\left.\begin{array}{l} \theta_k = \frac{\pi}{N} k \\ x_k = \cos(\theta_k) \end{array}\right\} \text{for } k = 0, \dots N$$

proof: $T_N(\theta) = \cos(N\theta)$

$$\frac{\partial T_N}{\partial \theta} = 0 = \sin(N\theta)$$

$$\therefore \theta = \frac{\pi}{N} k$$

$$(T_\alpha, T_\beta) = \sum_{k=0}^{N} w_k \cos(\alpha\theta_k)\cos(\beta\theta_k) = \begin{cases} 0 & \alpha \neq \beta \\ N & \alpha = \beta = 0, N \\ \frac{N}{2} & \alpha = \beta \neq 0 \end{cases}$$

insert w_k

$$w_k = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{1}{2} & k = N \\ 1 & \text{otherwise} \end{cases}$$

$$\sum_{k=0}^{N} w_k \cos(\alpha\theta_k)\cos(\beta\theta_k)$$

B. zeros sampling (based on zeros of T_{N+1})

$$\boxed{\begin{aligned}\theta_k &= \frac{(k+\frac{1}{2})\pi}{N+1} \quad k=0,..N \\ x_k &= \cos(\theta_k)\end{aligned}}$$

proof: $\cos((N+1)\theta)=0$

$(N+1)\theta = (2k+1)\frac{\pi}{2} \Rightarrow \theta = \frac{2k+1}{N+1}\frac{\pi}{2}$

$$\boxed{(T_\alpha, T_\beta) = \sum_{k=0}^{N} \cos(\alpha\theta_k)\cos(\beta\theta_k) = \begin{cases} 0 & \alpha \neq \beta \\ \frac{N+1}{2} & \alpha=\beta\neq 0 \\ N+1 & \alpha=\beta=0 \end{cases}}$$

insert

$w_k = 1 \quad \forall k$

example Method of weighted residual (MWR)

$$f(x) = \sum_{n=0}^{\infty}{}' a_n T_n(x) \qquad \boxed{-1 \le x \le 1}$$

if truncate the sum to N

the residual R

$$R = f(x) - \sum_{n=0}^{N}{}' a_n T_n(x) = \sum_{n=N+1}^{\infty}{}' a_n T_n(x)$$

The residual is equal to the unmodeled contribution of the sum

Finding a_n

Collocation (minimize residual at selected pts)

$$\min \int_{-1}^{1} R\,\delta(x - x_i)\,dx = 0 \qquad \text{minimize residual at a point } x_i$$

for each x_i

$$\boxed{f(x_i) - \sum_{n=0}^{N}{}' a_n T_n(x_i) = 0 \quad \text{for } x_0, x_1, \dots x_N}$$

$$\underline{f} = [T]\,\underline{a}$$

where $\underline{f} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix}$ $\quad T = \begin{bmatrix} T_0(x_0) & T_1(x_0) & \cdots & T_N(x_0) \\ & & & \\ T_0(x_N) & & & T_N(x_N) \end{bmatrix} = T_{ij}$

where $T_{ij} = T_j(x_i)$

$$\underline{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_N \end{bmatrix}$$

Diagonalization

Case 1: Extremal Sampling

$$\theta_k = \frac{\pi k}{N}$$

$k = 0, N$

given $\underline{f} = T\,\underline{a}$

One can diag the matrix T by premult.

by $[S]^T$

where

$$S = \begin{bmatrix} T_0(x_0)/2 & T_1(x_0)/2 & \cdots & T_N(x_0)/2 \\ T_0(x_1) & T_1(x_1) & & T_N(x_1) \\ T_0(x_2) & & & \\ T_0(x_{N-1}) & & & T_N(x_N)/2 \\ T_0(x_N)/2 & T_1(x_N)/2 & & \end{bmatrix}$$

S is just the T matrix with the last and first rows multiplied by 1/2

Then

$$\underbrace{[S]^T \underline{f}}_{\underline{F}} = \underbrace{[S]^T[T]}_{\text{diagonal}}\,\underline{a}$$

$$= S^T T = \begin{bmatrix} N & & & \\ & N/2 & & \\ & & \ddots & \\ & & N/2 & \\ & & & N \end{bmatrix}$$

The k^{th} component of $\underline{F}$

$$F_k = \frac{1}{2} f(x_0)\underbrace{T_K(x_0)}_{\cos\left(\frac{k0\pi}{N}\right)} + \sum_{n=1}^{N-1} f(x_n)\cos\left(\frac{k\pi}{N}n\right) + \frac{1}{2} f(x_N)\cos\left(\frac{k\pi}{N}N\right)$$

$$\boxed{F_k = \frac{1}{2}\left[f(x_0) + (-1)^k f(x_N)\right] + \sum_{n=1}^{N-1} f(x_n)\cos\left(\frac{k\pi}{N}n\right)}$$

DCT-I $\quad k = 0, N$

$$a_k = \frac{F_k}{\Lambda_k} \quad \text{where } \Lambda_k = \begin{cases} N & k = 0, N \\ N/2 & \text{otherwise} \end{cases}$$

$k=0$ or $k=N$

DCT-I of f

Case 2 : Zero sampling

$$\theta_k = \frac{2k+1}{2(N+1)}\pi \qquad x_k$$

$$\underline{f} = T\,\underline{a}$$

$$\underbrace{T^T \underline{f}}_{\underline{F}} = T^T T\,\underline{a} \qquad [T^T T] = \begin{bmatrix} N+1 & & & \\ & (N+1)/2 & & \\ & & \ddots & \\ & & & \frac{(N+1)}{2}\ N \end{bmatrix}$$

$$F_k = {\sum_{n=0}^{N}}' T_k(x_n) f(x_n) = \underbrace{{\sum_{n=0}^{N}}' f(x_n) \cos\left[\frac{2k+1}{2(N+1)}\pi n\right]}_{\text{DCT-II}}$$

Soln

$$a_k = \frac{F_k}{\Lambda_k} \qquad \text{where } \Lambda_k = \begin{cases} N+1 & k=0, N \\ \frac{N+1}{2} & \text{otherwise} \end{cases}$$

x_k or θ_k

Mapping example

$$\frac{d^2u}{d\hat{x}^2} = \alpha^2 u \qquad \text{bc} \quad u(0)=1, \quad u(\infty)=0$$

soln: $u = e^{-\alpha \hat{x}}$

mapping cheb-domain to physical-domain

cheb-dom physical-dom

the map $\hat{x} = \dfrac{1+x}{1-xa}$

x	$\hat{x}$
0	1
-1	0
1	$\frac{2}{1-a}$

← a is adjusted to set maximum physical range to be modeled

$$\frac{d^2 u}{d\hat{x}^2} = \alpha \ldots$$

Change of variable

$$\frac{du}{d\hat{x}} = \frac{dx}{d\hat{x}}\frac{du}{dx} \qquad \frac{d\hat{x}}{dx} = \frac{d}{dx}\left[\frac{1+x}{1-xa}\right] = \frac{a+1}{(ax-1)^2}$$

$$\boxed{\frac{du}{d\hat{x}} = \frac{(ax-1)^2}{a+1}\frac{du}{dx}}$$

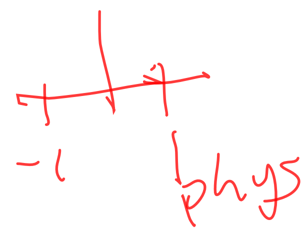
$$\frac{d^2u}{d\hat{x}^2} = \frac{dx}{d\hat{x}}\frac{d}{dx}\left(\frac{du}{d\hat{x}}\right) = \frac{dx}{d\hat{x}}\frac{d}{dx}\left[\frac{(ax-1)^2}{a+1}\frac{du}{dx}\right]$$

$$\frac{d^2u}{d\hat{x}^2} = \frac{d^2u}{dx^2}\underbrace{\frac{(ax-1)^4}{(a+1)^2}}_{\hat{\alpha}(x)} + \underbrace{\frac{2a(ax-1)}{(a+1)^2}}_{\hat{\beta}(x)}\frac{du}{dx}$$

$$\hat{\alpha}(x)u'' + \hat{\beta}(x)u' + \alpha^2 u = 0$$

bc $u(-1) = \cancel{\phi}$

$u(1) = 0$



$u(-1) \Rightarrow u(a)$

$u(1) \Rightarrow u(\infty)$

$$u = \sum_{n=0}^{N} c_n T_n(x)$$

$$\theta_i = \frac{\pi}{N} i$$

$$x_i = \cos(\theta_i)$$

$\theta_0 \Rightarrow x_0 = 1$

$\theta_N \Rightarrow x_N = -1$

$x_0 = +1$

$\underline{k=0}$ (bc)

$u(\infty)$ $0 = \sum' c_n T_n(x_0)$

$k=1, N-1$ $0 = \sum' c_n \left[\hat{\alpha}(x_k) T_n''(x_k) + \hat{\beta}(x_k) T_n'(x_k) + \alpha^2 T_n(x_k) \right]$

$k = N$ (bc)

$u(0)$ $1 = \sum_n' c_n T_n(x_N)$

$x_N = 1$