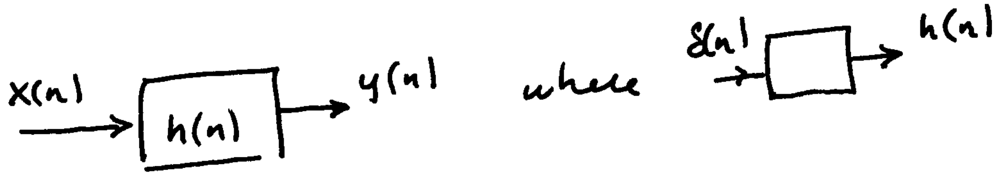
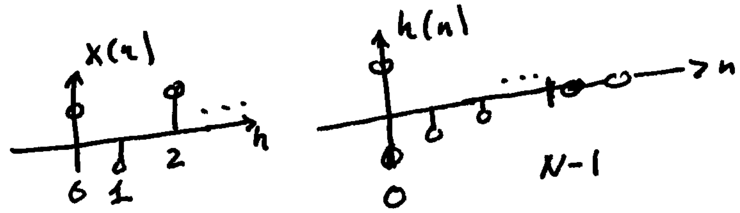
# SISO LTI




h(n) Finite impulse response $n = (0, N-1)$

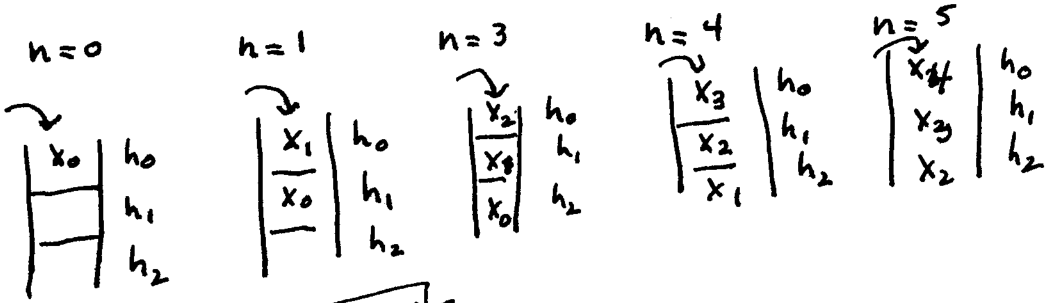
x(n) infinite duration $n = (0, \infty)$

$$y(n) = \sum_{k \in \infty} \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \equiv h * x$$

since $h(k) = 0 \mid 0 \geq k > N-1$

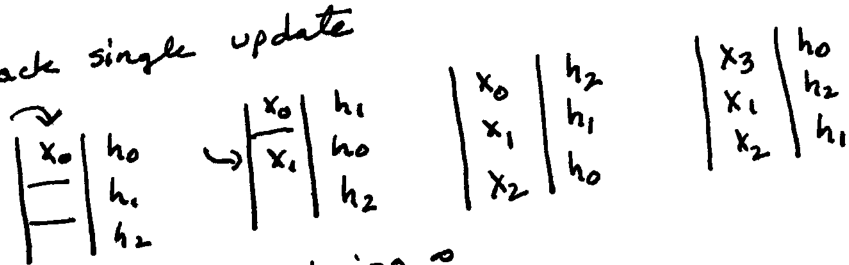
$$\boxed{y(n) = \sum_{k=0}^{N-1} h(k)\, x(n-k)}$$

stack approx $N=3$



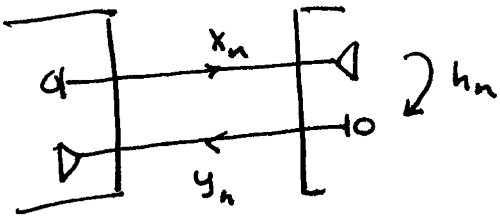

```
x = getx
push x of stack
y = stack_i · h_i
output y
```

Stack single update



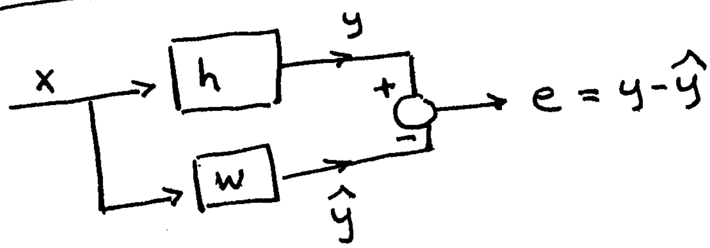

```
do j=0,∞
x = getx
stack(j % 3) = x
    do k=0,N-1
        y = y + stack(k) * h(((N-k+j) % 3)
    enddo
enddo
```

## Echo canceller

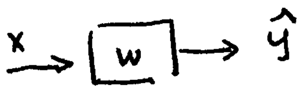


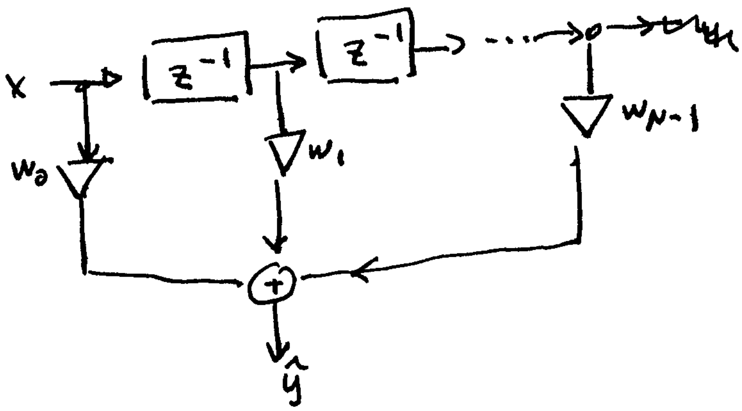

so $\quad y_n = x * h$

### objective




find transversal filter $w$
s.t. $e$ is minimized



<u>buffer</u>

$$\underline{x} = \begin{bmatrix} x_n \\ \vdots \\ x_{n-(N-1)} \end{bmatrix} \Rightarrow \begin{bmatrix} stk_0 \\ stk_1 \\ \\ stk_{N-1} \end{bmatrix}$$

<u>weights</u>

$$\underline{w} = \begin{bmatrix} w_0 \\ w_1 \\ \\ w_{N-1} \end{bmatrix}$$

$$\boxed{\begin{aligned} \hat{y}(n) &= \underline{w}^T \underline{x} \\ &= \begin{bmatrix} w_0 & w_1 & & w_{N-1} \end{bmatrix} \begin{bmatrix} stk_0 \\ stk_1 \\ \\ stk_{N-1} \end{bmatrix} \end{aligned}}$$

The error

$$e_n = y_n - \hat{y}_n$$

where $\hat{y}_n = \underline{w}^T \underline{x}$

The mean sq error

$$m_n = E(e_n e_n)$$

$$e = y - \hat{y}$$

$$e^2 = y^2 + \hat{y}^2 - 2y\hat{y}$$

$$= y^2 + (\underline{w}^T\underline{x})(\underline{w}^T\underline{x}) - 2y(\underline{w}^T\underline{x})$$

$$\boxed{e^2 = y^2 + \underline{w}^T(\underline{x}\,\underline{x}^T)\underline{w} - 2\underline{w}^T(y\underline{x})}$$

$$E(e^2) = E(y^2) + \underline{w}^T E(\underline{x}\,\underline{x}^T)\underline{w} - 2\underline{w}^T E(y\underline{x})$$

define

$$E(y\underline{x}) \equiv \underline{P}$$

$$E(y^2) = \sigma_y^2$$

$$E(y) = 0$$

$$[R] = E(\underline{x}\,\underline{x}^T)$$

$$\boxed{m = E(e^2) = \sigma_y^2 - 2\underline{w}^T\underline{P} + \underline{w}^T R\underline{w}}$$

<u>minimize m</u> find $\underline{w}$

$$\frac{dm}{d\underline{w}} = 0 = -2\underline{P} + R\underline{w} + R^T\underline{w}$$

where $R = R^T$

$$\boxed{R\underline{w}_{opt} = \underline{P}}$$

## Solution by iteration

### gradient descent

$$\underline{w}_{new} = \underline{w}_{old} - \mu \left[\frac{dm}{d\underline{w}}\right]$$

$$\underline{w}_{new} = \underline{w}_{old} - \mu \left[-2\underline{P} + 2R\underline{w}_{old}\right]$$

### instantaneous est

$$R = \underline{x}\,\underline{x}^T$$

$$\underline{P} = \underline{x}\,y$$

$$\left[-2P + 2R\underline{w}_{old}\right] = -2\underline{x}y + 2\underline{x}\,\underline{x}^T\underline{w}_{old}$$

$$= 2\underline{x}\,\underbrace{\left[-y + \underline{x}^T\underline{w}_{old}\right]}_{-e}$$

$$= -2xe$$

$$\boxed{\underline{w}_{new} = \underline{w}_{old} + 2\mu\,\underline{x}\,e}$$

convergence

$$0 < \mu < \frac{2}{N\sigma_x^2}$$

**NLMS** (norm L.S)

$$\underline{w}_{new} = \underline{w}_{old} + \beta\left(\frac{2}{N\sigma_x^2}\right)\underline{x}\,e$$

$$0 < \beta < 1$$

Approx. Theory

BG5

## Orthogonal Function

consider the set of function $\{f_i\}$ defined $a < x < b$; $i = 1, N$

$$(f_i, f_j) \equiv \int_a^b w(t)\, f_i(t)\, f_j(t)\, dt = \begin{cases} 0 & i \neq j \\ \Lambda_i & i = j \Leftarrow \text{orthogonal} \\ 1 & i = j \Leftarrow \text{orthonormal} \end{cases}$$

$w(t)$ ← weight function

## Application of orthogonality

Consider
$$F(x) = \sum_{n=1}^{\infty} a_n f_n(x) \quad a < x < b$$

where $\{f_n\}$ are ortho.

Find $a_n$

$$w(x) f_k(x) F(x) = \sum_{n=1} a_n\, w(x) f_k(x) f_n(x)$$

$$\int_a^b w f_k F\, dx = \sum_{n=1} a_n \int_a^b w f_k f_n\, dx$$

since $\int_a^b w f_k f_n = \begin{cases} \Lambda_n & k = n \\ 0 & k \neq n \end{cases}$

$$\int_a^b w f_n F dx = a_n \int_a^b w f_n^2 dx$$

$$\boxed{a_n = \frac{\int_a^b w f_n F dx}{\int_a^b w f_n^2 dx}}$$

Least-sq

$$F(x) = \sum_{j=1}^{M} c_j\, g_j(x)$$

the residual

$$\epsilon(x) = F(x) - \sum_{j=1}^{M} c_j\, g_j(x)$$

$$\epsilon^2(x) = \left[F - \sum_{j=1}^{M} c_j g_j(x)\right]^2$$

$$\cancel{w\,E(x)})\quad E = \int_a^b w\epsilon^2\,dx = \int_a^b w\left[F - \sum_{j=1}^{M} c_j g_j\right]^2 dx$$

$$E = \int_a^b \left[ wF^2 - 2\sum_{j=1}^{M} c_j\, F g_j w + \sum_{j=1}^{M}\sum_{\ell=1}^{M} c_j c_\ell\, g_j g_\ell w \right] dx$$

minimize $E$

$$\frac{\partial E}{\partial c_k} = 0 = \int_a^b \left[ -F g_k w 2 + \sum_{\ell=1} 2 c_\ell\, g_k g_\ell w \right] dx \quad \text{for } k = 1, M$$

$$\sum_{\ell=1}^{M} c_\ell \left[ \int_a^b g_k g_\ell w\,dx \right] = \int_a^b F g_k w\,dx$$

$$\begin{bmatrix} a_{11} & a_{12} & \\ & a_{ij} & \\ & & a_{mm} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_M \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_M \end{bmatrix}$$

where

$$a_{ij} = \int_a^b g_i g_j w\,dx$$

$$b_i = \int_a^b F g_i w\,dx$$

if $\{g_i\}$ are orthogonal function

$$a_{ij} = \begin{cases} 0 & i \neq j \\ -\lambda_i & i = j \end{cases}$$

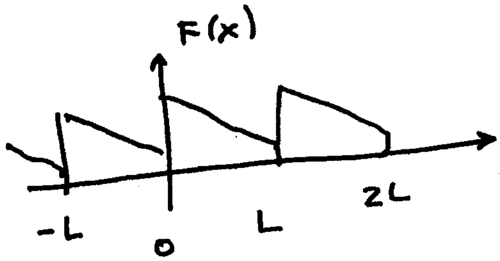
$$\begin{bmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

therefore

$$c_i = \frac{\int_a^b g_i F w \, dx}{\int_a^b w g_i^2 \, dx}$$

orthogonal funct min. the error in the least sq sense

## Continuous Fourier Expansion




$F(x)$ periodic

$F(x) = F(x + kL) \quad k = 0, \pm 1, \pm 2 \ldots$

$$F(x) = \sum_{n=-\infty}^{\infty} a_n \phi_n(x)$$

where $\phi_n(x) = e^{i\frac{2\pi}{L} n x}$

<u>orthogonality</u>

$$\int_0^L \phi_n \phi_m^* \, dx = \int_0^L e^{i\frac{2\pi}{L}(n-m)x} dx = \begin{cases} 0 & n \neq m \\ L & n = m \end{cases}$$

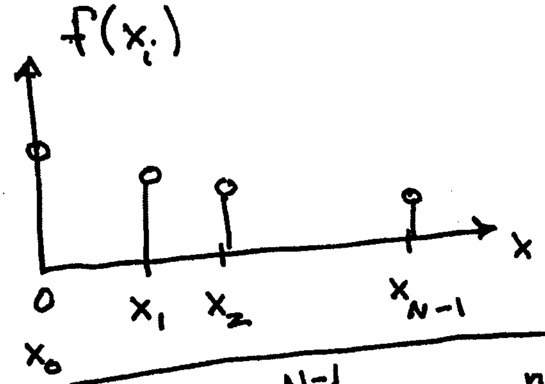
<u>then</u>

$$\phi_m^* F = \sum_{n=-\infty}^{\infty} a_n \phi_n \phi_m^*$$

$$\int_0^L \phi_m^* F dx = \sum_{n=-\infty}^{\infty} a_n \int_0^L \phi_n \phi_m^* dx$$

$$\boxed{a_n = \frac{\int_0^L \phi_n^* F(x) dx}{L}}$$

# Discrete Fourier Series (DFS)




where

$$x_i = i\left[\frac{2\pi}{N}\right] = i\,dx$$

$$dx = \frac{2\pi}{N}$$

$$\Rightarrow \boxed{f(x_i) = \frac{1}{N}\sum_{n=0}^{N-1} F(n)\, W_i^{\,n}}$$

where $W_i = e^{jx_i}$

$x_i = i\left[\frac{2\pi}{N}\right]$

$i = (0, \ldots N-1)$

$j = \sqrt{-1}$

## Find F(n)

$$f(x_i) = \frac{1}{N}\sum_{n=0}^{N-1} F(n)\, W_i^{\,n} \qquad \text{step 1}$$

$$(W_i^{\,k})^* f(x_i) = \frac{1}{N}\sum_{n=0}^{N-1} F(n)\, W_i^{\,n} (W_i^{\,k})^* \qquad \text{step 2}$$

mult by complex conjugate $(W_i^{\,k})^*$

$$\sum_{i=0}^{N-1} (W_i^{\,k})^* f(x_i) = \frac{1}{N}\sum_{n=0}^{N-1} F(n)\left[\sum_{i=0}^{N-1} W_i^{\,n}(W_i^{\,k})^*\right] \qquad \text{step 3}$$

sum over $i$

Focus on the

~~The~~ bracketed term

$$\left[\sum_{i=0}^{N-1} W_i^n \left(W_i^k\right)^*\right] = \sum_{i=0}^{N-1} e^{j x_i (n-k)} = \boxed{\sum_{i=0}^{N-1} e^{j \frac{2\pi}{N}(n-k) i}}$$

geometric series

$$= \frac{1 - e^{j 2\pi (n-k)}}{1 - e^{j \frac{2\pi}{N}(n-k)}} = \begin{cases} N & \text{if } n-k=0 \\ 0 & \text{if } n-k\neq 0 \end{cases}$$

$$= N\delta_{n-k} = \begin{cases} N & n-k=0 \\ 0 & n-k\neq 0 \end{cases}$$

Step 4

$$\sum_{i=0}^{N-1} f(x_i)\left(W_i^k\right)^* = \frac{1}{N}\sum_{n=0}^{N-1} F(n) \left(N\delta_{n-k}\right)$$

$$\boxed{\sum_{i=0}^{N-1} f(x_i)\left(W_i^k\right)^* = F(k)}$$

Final

$$f(x_i) = \frac{1}{N}\sum_{k=0}^{N-1} F(k)\, W_i^k \qquad i = (0 \ldots N-1)$$

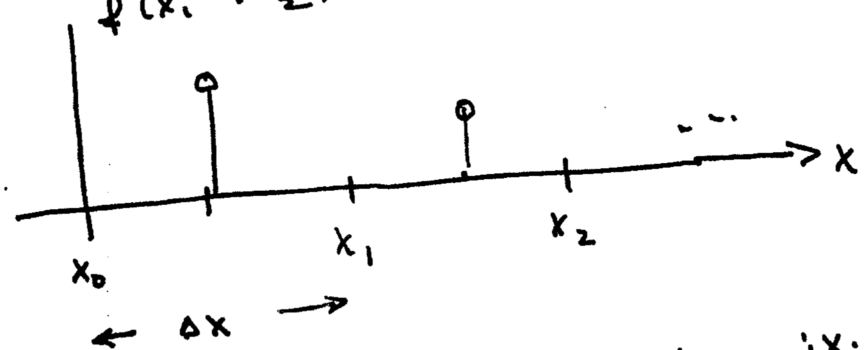
$$F(k) = \sum_{i=0}^{N-1} f(x_i)\left(W_i^k\right)^* \qquad k = (0, \ldots N-1)$$

$$W_i^k = e^{j x_i k} \qquad x_i = \frac{2\pi}{N}(i)$$

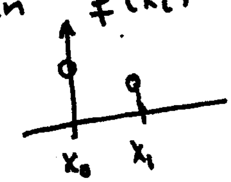
$$\boxed{f(x_i) \longleftrightarrow F(k)}$$

**Example** given $f(x_i) \leftrightarrow F(k)$

Find $f\left(x_i + \frac{\Delta x}{2}\right)$




given




$$f(x_i) = \frac{1}{N} \sum_{n=0}^{N-1} F(n)\, e^{j x_i n}$$

then
$$f\left(x_i + \frac{\Delta x}{2}\right) = \frac{1}{N} \sum_{n=0}^{N-1} \underline{\underline{F(n)}}\, e^{j x_i n}\, \underline{\underline{e^{j \frac{\Delta x}{2} n}}}$$

$$\boxed{f\left(x_i + \frac{\Delta x}{2}\right) \leftrightarrow F(k)\, e^{j \frac{\Delta x}{2} n}}$$

## Using the FFT to obtain $f(x_i + \frac{\Delta x}{2})$

**step 1** $f(x_i) \xrightarrow[\text{FFT}]{} F(k)$

**step 2** compute phase

$$C(k) = e^{j\pi k/N} \qquad k = 0, N/2$$

$$C(N-k) = \text{conjugate}(C(k)) \qquad k = 1, N/2$$

**step 3**

$$F_{new}(k) = F(k)C(k)$$

**step 4**

$$F_{new}(k) \xrightarrow[\text{inverse FFT}]{} f(x_i + \Delta x)$$

(B) Given $F(k)$

find $\frac{df}{dx}$

step 1 $f \xrightarrow[\text{FFT}]{} F$

step 2 compute weight factor

$C(k) = jk$ for $k = 0, N/2$

$C(k) = j(N-k)$ for $k = \frac{N}{2}+1, N-1$

step 3 $F_{new}(k) = F(k)C(k)$

step 4 $F_{new}(k) \xrightarrow[\text{IFFT}]{} \frac{df}{dx}$

proof: $f = \frac{1}{N}\sum_{n=0}^{N-1} F(n) e^{jx_i n}$

$\frac{df}{dx} = \frac{1}{N}\sum_{n=0}^{N-1} \left(jnF(n)\right) e^{jx_i n}$

(c) Product anti aliasing

$u(n) \rightarrow \otimes \rightarrow w(n) = u(n)v(n)$

$v(n) \nearrow$

$$u(n) = \frac{1}{N}\sum_{k=0}^{N-1} U_k\, W^{nk}$$

$$v(n) = \frac{1}{N}\sum_{k=0}^{N-1} V_k\, W^{nk}$$

$u, v$ and $U_k, V_k$, $W = e^{j\frac{2\pi}{N}}$ are known

Find the Fourier transform of $uv$

$$W_k = \sum_n{}' u(n)v(n)\, W^{-nk}$$

$$W_k = \sum_n{}' \sum_l{}' \sum_m V_m U_l\, W^{+n(l+m-k)} \frac{1}{N^2}$$

$$W_k = \sum_l \sum_m U_l V_m \left[\sum_{n=0} W^{n(l+m-k)}\right] \frac{1}{N^2}$$

where $[\ ] = \begin{cases} N & \text{for } l+m-k=0 \\ N & \text{for } l+m-k = \pm N \\ 0 & \text{otherwise} \end{cases}$

$$\boxed{W_k = \underbrace{\sum_l \sum_m{}' U_l V_m \frac{\delta_{l+m-k}}{N}}_{\text{unaliased}} + \underbrace{\sum_l \sum_m U_l V_m \frac{\delta_{l+m-k+N}}{N} + \sum_l{}' \sum_m{}' U_l V_m \frac{\delta_{l+m-k-N}}{N}}_{\text{aliased}}}$$

Aliasing removal

Define ½ step time series

$$\hat{u}(n) = \frac{1}{N} \sum_k{}' U_k \, W^{nk} \, W^{\frac{1}{2}k}$$

$$\hat{v}(n) = \frac{1}{N} \sum_k{}' V_k \, W^{nk} \, W^{\frac{1}{2}k}$$

$$\hat{W}_k = \sum_n{}' \hat{v}(n) \, \hat{u}(n) \, W^{-nk} \, W^{-k/2}$$

$$\hat{W}_k = \sum_l{}' \sum_m{}' U_l V_m \left[ \sum_n{}' W^{n(l+m-k)} \right] W^{(l+m-k)/2} \frac{1}{N^2}$$

$$\hat{W}_k = \sum_l \sum_m{}' U_l V_m \frac{\delta_{l+m-k}}{N} - \sum_l{}' \sum_m{}' U_l V_m \frac{\delta_{l+m-k+N}}{N} - \sum_l{}' \sum_m{}' U_l V_m \frac{\delta_{l+m-k-N}}{N}$$

$$\boxed{\tilde{W}_k = \frac{1}{2} \left[ W_k + \hat{W}_k W^{-k/2} \right]}$$

$\tilde{W}_k \xrightarrow[\text{IFFT}]{} w(n)$ unaliased

## Antialias $w = v \cdot u$

**step 1** Phase factor

$$C_1(k) = \begin{cases} e^{j\frac{2\pi}{2N}(k)} & k = 0, N/2 \\ e^{-j\frac{2\pi}{2N}(k)} & k = N/2+1, N-1 \end{cases}$$

$$C_2(k) = C_1^*(k)$$

**step 2**

$$u(n) \xrightarrow{\text{FFT}} U_k$$
$$v(n) \longrightarrow V_k$$

**step 3**

$$C_1(k)U_k \xrightarrow{\text{IFFT}} \hat{u}(n)$$
$$C_1(k)V_k \longrightarrow \hat{v}(n)$$

half step ver. of $u, v$

**step 4**

$$w = u \cdot v$$
$$\hat{w} = \hat{u} \cdot \hat{v}$$

**step 5**

$$w \xrightarrow{\text{FFT}} W_k$$
$$\hat{w} \longrightarrow \hat{W}_k$$

**step 6**

$$W_{k_{new}} \Rightarrow \frac{1}{2}\left[ W_k + C_2(k)\hat{W}_k \right]$$

**step 7**

$$\cancel{\text{Take}} \quad W_{k_{new}} \xrightarrow{\text{IFFT}} w_{new}$$