Lecture 1

① Linear Algebra

vectors — matricies

vector

(a) columnvector $\underline{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}$ v_i : i^{th} element

(b) row vector $\underline{v}^T = [v_1 \dots v_N]$

operations

① inner product

② outer product

ie inner product $(\underline{a}, \underline{b})$

given : $\underline{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix}$ $\underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix}$

$$s = \underline{a}^T \underline{b} = [a_1 \dots a_N] \begin{bmatrix} b_1 \\ b_N \end{bmatrix}$$

$$s = \sum_{i=1}^{N} a_i b_i = a_1 b_1 + a_2 b_2 \dots a_N b_N$$

ie outer product $\underline{a})(\underline{b}$ $\underline{a}\,\underline{b}^T$

$$A_{ij} = a_i b_j \Leftarrow \underline{a}\,\underline{b}^T = \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} [b_1 \dots b_N]$$

$$A = \begin{bmatrix} a_1 b_1 & & a_1 b_N \\ & & \\ a_N b_1 & & a_N b_N \end{bmatrix}$$

Prob Given two N-element vectors labeled $\underline{a}$, $\underline{b}$

find

(1) $(\underline{a}, \underline{b}) = s$ $\Longrightarrow s = a_i b_i$

(2) $\underline{a})(\underline{b} = [\hat{s}]$ $\Longrightarrow \hat{s}_{ij} = a_i b_j$

C-lang

```
s=0
(1)  for(i=0; i<N; i++){
     s=s+a[i]*b[i]; }

     for(i=0; i<N; i++{
       for(j=0; j<N; j++{
(2)      Shat[i][j] = a[i]*b[i]
       };
     }
```

Fortran

```
s=0
(1) do i=0,N-1
      s = s + a(i)*b(i)
    enddo
      do i=0,N-1
        do j=0,N-1
(2)       shat(i,j) = a(i)*b(j)
        enddo
      enddo
```

③ <u>Cumulative sum</u>

$$f_j = \sum_{i=1}^{j} g_i \quad \text{For } j = (1, N)$$

$$f_j = \begin{cases} g_1 & j = 1 \\ f_{j-1} + g_j & \cancel{j \geq 1}\ N > j > 1 \end{cases}$$

④ <u>Cumulative product</u>

$$f_j = \prod_{i=1}^{j} g_i \quad \text{For } j = 1, N$$

$$f_j = \begin{cases} g_1 & j = 1 \\ f_{j-1}\, g_j & N > j > 1 \end{cases}$$

Matrix

$$A = \begin{array}{c} \longrightarrow j \\ \downarrow i \end{array} \left[a_{ij} \right] \qquad \begin{array}{l} i = 1, N \\ j = 1, M \end{array}$$

Operation

① Equality

$$A = B \Rightarrow A_{ij} = B_{ij}$$

A_{ij}

② Addition

$$C = A + B \Rightarrow C_{ij} = A_{ij} + B_{ij}$$

③ Multiplication by scalar

$$C = \alpha A \Rightarrow C_{ij} = \alpha A_{ij}$$

④ Matrix vector multiply

$$\underline{v} = A\,\underline{x}$$

given $\underline{v} = N \times 1$

$\underline{x}$ $M \times 1$

$A = N \times M$

$$\Rightarrow v_i = \sum_{j=1}^{M} A_{ij}\, x_j \Rightarrow \text{step 1}$$

for $i = 1, N$

<u>step 1</u>

```
v_i = 0
do j = 1, M
  v_i = A_ij x_j + v_i
enddo
```

step 2

```
do i = 1, N
  [ ]
enddo
```

(5) Matrix - Matrix multiply

$$C = AB$$

where

C $N \times L$

B $M \times L$

$A = N \times M$

$$C_{ij} = \sum_{k=1}^{M} A_{ik} B_{kj}$$

Step 1

```
C_ij = 0
do k = 1, M
C_ij = C_ij + A_ik B_kj
enddo
```

Step 2

```
do i = 1, N
do j = 1, L
    [ ]
enddo
enddo
```

Algebric eqns

$$A\underline{x} = \underline{b}$$

given $\underline{b}$ & A find $\underline{x}$ and $A\,(N \times M)$ $\underline{b}\,(N \times 1)$
$\underline{x} = (M \times 1)$

① System types

① Determined $N = M$

② over-determined ~~$N < M$~~ $N > M$

③ under-determined ~~$M > N$~~ $N < M$

Determined system $N = M$

(a) Direct method

$$A\underline{x} = \underline{b} \Rightarrow \underline{x} = A^{-1}\underline{b}$$

$$\underline{x} = \frac{1}{|A|}\left[\text{adjoint } A\right]\underline{b}$$

note:
1. determining the adjoint requires $\mathcal{O}(N\,(N+1)!)$ ops

2. $|A| \to 0$, $\frac{1}{|A|} \to \infty$ } ~~small variation in b~~ in such a case what is $\underline{x}$?

Evaluating the sensitivity of $\underline{x}$ to $\underline{b}$

norms

vector

$$\|\underline{x}\|_p = \left\{ \sum_{i=1}^{N} |x_i|^p \right\}^{1/p} \quad \text{p-norm}$$

ie 2-norm $\|\underline{x}\|_2 = \left\{ \sum_{i=1}^{N} |x_i|^2 \right\}^{1/2}$

∞-norm $\|\underline{x}\|_\infty = \max |x_i|$

1-norm $\|\underline{x}\|_1 = \sum_{i=1}^{N} |x_i|$

matrix

$$\|A\|_p = \max_{1 \le i \le N} \left\{ \sum_{j=1}^{N} |a_{ij}|^p \right\}^{1/p}$$

ie $\|A\|_1$ = max column mag sum

$\|A\|_\infty$ = max row mag sum

$\|A\|_2$ = max eigenvalue AA^T

$A\underline{x} = \underline{z}$

$\|A\underline{x}\| = \|\underline{z}\|$, $|A\underline{x}|$

ex

$$A = \begin{bmatrix} 1 & -2 \\ -3 & -4 \end{bmatrix}$$

$\|A\|_1$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

4 6

$$\|A\|_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$\downarrow$ $\downarrow$

4 6

$$\|A\|_\infty = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{matrix} \rightarrow 3 \\ \rightarrow 7 \end{matrix}$$

$$\|A\|_2 \Rightarrow AA^T = \begin{bmatrix} 1 & -2 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -2 & -4 \end{bmatrix} = \begin{bmatrix} 5 & -11 \\ -11 & 25 \end{bmatrix}$$

$$|AA^T - \lambda I| \Rightarrow \begin{cases} \lambda = 15 - \sqrt{221} \\ \boxed{\lambda = \sqrt{221} + 15} \end{cases}$$

$$A = \begin{bmatrix} 1 & -2 \\ -3 & -4 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & -3 \\ -2 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -2 & -4 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 16 \end{bmatrix}$$

$-3+8$ $\quad -3+8$ $\quad 9+16$

$$\left[AA^T - \lambda I\right] = \begin{bmatrix} 5-\lambda & 5 \\ 5 & 16-\lambda \end{bmatrix}$$

$$(5-\lambda)(16-\lambda) - 25 = 0$$

λ

Condition number (measurement of sensitivity of the soln to changing input)

Consider

$$A\underset{\text{soln}}{\underline{x}} = \underset{\text{input}}{\underline{b}}$$

$$A\tilde{\underline{x}} = \underline{b} + \underline{r}$$

$$A[\underline{x} - \tilde{\underline{x}}] = \underline{r}$$

$$\boxed{\underline{e} = \underline{x} - \tilde{\underline{x}} \quad \text{error}}$$

$$\boxed{A\underline{e} = \underline{r}}$$

$$\underline{e} = A^{-1}\underline{r}$$

$$\underline{r} = A\underline{e}$$

$$\|\underline{r}\| = \|A\underline{e}\|$$

$$\|\underline{r}\| \le \|A\| \, \|\underline{e}\|$$

$$\|\underline{e}\| = \|A^{-1}\underline{r}\|$$

$$\|\underline{e}\| \le \|A^{-1}\| \, \|\underline{r}\|$$

note $\|A\underline{x}\| \le \|A\| \, \|\underline{x}\|$

$$\|\underline{e}\| \le \|A^{-1}\| \, \|\underline{r}\|$$

$$\frac{\|\underline{r}\|}{\|A\|} \le \|\underline{e}\|$$

$$\boxed{\frac{\|\underline{r}\|}{\|A\|} \le \|\underline{e}\| \le \|A^{-1}\| \, \|\underline{r}\|}$$

$$\|\underline{r}\| \le \|A\| \, \|\underline{e}\|$$

$$\frac{\|\underline{r}\|}{\|A\|} \le \underline{e}$$

error is bracketed

Divide by $\|\underline{x}\|$ norm of the true solut

$$\frac{\|\underline{r}\|}{\|A\|\,\|\underline{x}\|} \le \frac{\|\underline{e}\|}{\|\underline{x}\|} \le \|A^{-1}\| \frac{\|\underline{r}\|}{\|\underline{x}\|}$$

since

$$\underline{x} = A^{-1}\underline{b}$$

$$|\underline{x}| \le |A^{-1}|\,|\underline{b}|$$

$$\boxed{\frac{1}{|\underline{x}|} \ge \frac{1}{|A^{-1}|\,|\underline{b}|}}$$

$$\underline{b} = A\underline{x}$$

$$|\underline{b}| \le |A|\,|\underline{x}|$$

$$\frac{|\underline{b}|}{|A|} \le |x|$$

$$\boxed{\frac{1}{|\underline{x}|} < \frac{|A|}{|\underline{b}|}}$$

$$\frac{\|\underline{r}\|}{|A^{-1}|\,|A|\,|\underline{b}|} \le \frac{r}{\|A\|\,\|\underline{x}\|} < \frac{|\underline{e}|}{|x|} \le \frac{|A^{-1}|\,|\underline{r}|}{|x|} < \frac{|A|\,|A^{-1}|\,|r|}{\underline{b}}$$

Condition number

cond $\|A\|\,\|A^{-1}\|$

$$\boxed{\frac{1}{\text{cond}(A)}\frac{|\underline{r}|}{|\underline{b}|} < \frac{|\underline{e}|}{|x|} < \frac{|r|}{|\underline{b}|}\,\text{cond}(A)}$$

Direct Method of soln $A\underline{x} = \underline{b}$

Elimination

① $\underbrace{TA}\underline{x} = T\underline{b} = \hat{b}$

$U\underline{x} = \hat{\underline{b}}$ $\qquad (TA) = U \quad \mathcal{O}(N^3)$

$U = \left[\begin{matrix} \diagdown & \\ 0 & \diagdown \end{matrix}\right]$ ✓

Back subst. ② $\cancel{b_i = \sum_{j=} U_{ij}}$ $U\underline{x} = \hat{\underline{b}}$

$$\boxed{\hat{b}_i = \sum_{j=i}^{N} U_{ij} x_j}$$

$$\hat{b}_i = \sum_{j=i}^{N} U_{ij} x_j$$

$j = i$

$$\hat{b}_N = U_{NN} x_N$$

$$\hat{b}_{N-1} = U_{N-1,N-1} x_{N-1} + U_{N-1,N} x_N$$

$$\hat{b}_i = U_{ii} x_i + \sum_{j=i+1}^{N} U_{ij} x_j$$

$$\boxed{\begin{aligned} x_N &= \frac{\hat{b}_N}{U_{NN}} \\ x_i &= \frac{\hat{b}_i}{U_{ii}} - \sum_{j=i+1}^{N} U_{ij} x_j \frac{1}{U_{ii}} \end{aligned}}$$

$\Rightarrow \mathcal{O}(N^2)$

Decomposition

① LU

$$A = [L \backslash 0][0 \backslash U]$$

$$\boxed{A\underline{x} = \underline{b}} \Rightarrow \boxed{LU\underline{x} = \underline{b}} \begin{cases} \underline{y} = U\underline{x} \\ L\underline{y} = \underline{b} \end{cases}$$

soln

① $L\underline{y} = \underline{b}$ find $\underline{y}$ by forward sub

② $U\underline{x} = \underline{y}$ find $\underline{x}$ by backward sub

~~when~~ as $|A| \to 0$ round-off error

$$LU\underline{x} = \underline{b}$$

$$\boxed{\begin{aligned} L[\underline{y}] &= \underline{b} \\ \underline{y} &= U\underline{x} \end{aligned}}$$

$\Rightarrow$ $\underline{y}$ by forward

$\leftarrow$ find $\underline{x}$ backward

$QQ^T = I$

② QR decomp

$A\underline{x} = \underline{b}$

$\boxed{A = QR}$ where $QQ^T = I$

Q is orthogonal matrix

① $QR\underline{x} = \underline{b} = \begin{cases} \underline{y} = R\underline{x} & \Rightarrow \underline{x} \text{ backsub} \\ Q\underline{y} = \underline{b} \Rightarrow \boxed{\underline{y} = Q^T\underline{b}} \end{cases}$

② lower cond

$$A = \begin{bmatrix} Q \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix}$$

$A\underline{x} = \underline{b}$

$QR\underline{x} = \underline{b}$

$$\boxed{\begin{aligned} Q[\underline{y}] &= \underline{b} \\ \underline{y} &= R\underline{x} \end{aligned}}$$

Because Q is unitary matrix

$Q^TQ[\underline{y}] = Q^T\underline{b}$

$I\,\underline{y} = Q^T\underline{b}$

Over determined system $N > M$

$$\begin{array}{c} \rightarrow M \leftarrow \\ \downarrow \\ N \; A \\ \uparrow \end{array}$$

$$A\underline{x} = \underline{b}$$

least sq

$$\underline{\epsilon} = [A\underline{x} - \underline{b}]$$

$\underline{\epsilon}$ is the error vector

$$\underline{\epsilon}^T\underline{\epsilon} = (A\underline{x} - \underline{b})^T(A\underline{x} - \underline{b})$$

$$= [\underline{x}^T A^T - \underline{b}^T][A\underline{x} - \underline{b}]$$

$$E = \underline{x}^T A^T A\underline{x} - \underline{x}^T A^T b - \underline{b}^T A\underline{x} + \underline{b}^T\underline{b}$$

minimized E via $\underline{x}$ selection

$$\frac{\partial E}{\partial \underline{x}} = 0 = A^T A\underline{x} + (\underline{x}^T A^T A)^T - (A^T\underline{b}) - (\underline{b}^T A)^T + 0$$

$$0 = A^T A\underline{x} + A A^T\underline{x} - A^T\underline{b} - A^T\underline{b}$$

$$0 = 2A^T A\underline{x} - 2A^T\underline{b}$$

$$\boxed{A^T A\underline{x} = A^T\underline{b}}$$

normal eqns

Singular Value Decomposition method for overdetermined systems

① $A = U \Sigma' V^T$ where A $(N \times M)$

$$\left.\begin{array}{l} U \text{ is } N \times N \\ V \text{ is } M \times M \\ \Sigma' = N \times M \end{array}\right\} \quad A = \begin{bmatrix} U \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_M \\ \hline & & 0 & \end{bmatrix} \begin{bmatrix} V \end{bmatrix}^T$$

U and V are unitary matrixes

$U^T U = I_{N \times N}$; $V^T V = I_{M \times M}$

Consider

$$\underline{\epsilon} = A\underline{x} - \underline{b}$$

$$\underline{\epsilon} = U \Sigma V^T \underline{x} - \underline{b}$$

$$\underbrace{U^T \underline{\epsilon}}_{\hat{\underline{\epsilon}}} = \Sigma' \underbrace{V^T \underline{x}}_{\hat{\underline{x}}} - \underbrace{U^T \underline{b}}_{\hat{\underline{b}}}$$

$$\boxed{\hat{\underline{\epsilon}} = \Sigma' \hat{\underline{x}} - \hat{\underline{b}}}$$

$$E = \hat{\underline{e}}^T \underline{e} = \left[\Sigma \hat{\underline{x}} - \hat{\underline{b}}\right]^T \left[\Sigma \hat{\underline{x}} - \hat{\underline{b}}\right]$$

$$E = \sum_i \left[\sigma_i \hat{x}_i - b_i\right]^2$$

<u>minimize</u>

$$\frac{\partial E}{\partial \hat{x}_i} = 0 = 2\left[\sigma_i \hat{x}_i - \hat{b}_i\right]\sigma_i = 0$$

$$\boxed{\text{solution} \quad \hat{x}_i = \begin{cases} 0 & \text{if } \sigma_i = 0 \\ \dfrac{\hat{b}_i}{\sigma_i} & \text{if } \sigma_i \neq 0 \end{cases}}$$

$$\boxed{\hat{\underline{x}} = V^T \underline{x}}$$

$$\boxed{\underline{x} = V \hat{\underline{x}}}$$

$$\underline{x} V^T = \hat{\underline{x}}$$

$$V^T V = I$$

$$\underline{x} = V \hat{\underline{x}}$$

$$\sigma \begin{bmatrix} 1 \\ 10^{-2} \\ 10^{-17} \end{bmatrix}$$

$$\hat{x}_1 = \frac{\hat{b}_1}{1} \quad \checkmark \qquad \hat{x}_2 = \frac{\hat{b}_2}{10^{-2}} \quad \checkmark \qquad \hat{x}_3 = \frac{\hat{b}_3}{10^{-17}}$$

$$\hat{x}_3 = 0$$

Underdetermined $N < M$

Mininorm if soln which is closest to the origin

Find $\underline{x}$ such that $|\underline{x}|^2$ is minimized subject to the constraint $A\underline{x} - \underline{b} = 0$

$$\boxed{V = \underline{x}^T \underline{x} + \underline{\lambda}^T (A\underline{x} - \underline{b})}$$

First variation

$$\delta V = \left(\frac{\partial V}{\partial \underline{x}}\right)^T \cdot \delta \underline{x} + \left(\frac{\partial V}{\partial \underline{\lambda}}\right)^T \cdot \delta \underline{\lambda} = 0$$

Find the gradient $\frac{\partial V}{\partial \underline{x}}$ & $\frac{\partial V}{\partial \underline{\lambda}}$

$\delta \underline{x}$ & $\delta \underline{\lambda} \neq 0$

$$\frac{\partial V}{\partial \underline{x}} = 0$$

$$\frac{\partial V}{\partial \underline{\lambda}} = 0$$

$$\frac{\partial V}{\partial \underline{x}} = \underline{x} + (\underline{x}^T)^T + (\underline{\lambda}^T A)^T$$

$$\frac{\partial V}{\partial \underline{x}} = 2\underline{x} + A^T \underline{\lambda} \Rightarrow \boxed{\underline{x} = -\frac{1}{2} A^T \underline{\lambda}} \quad (1)$$

$$\frac{\partial V}{\partial \underline{\lambda}} = A\underline{x} - \underline{b} = 0 \Rightarrow \boxed{A\underline{x} - \underline{b} = 0} \quad (2)$$

---

let $\underline{q} = -\frac{\underline{\lambda}}{2}$

$$\boxed{\underline{x} = A^T \underline{q}}$$

$$\cancel{(1) \to (2)} \quad \cancel{-\frac{1}{2} A^T \underline{A} - \underline{b} = 0}$$

~~let $\underline{q} = -\frac{\underline{\lambda}}{2}$ then~~

~~$\underline{x} =$~~

$$A\underline{x} - \underline{b} = 0 \Rightarrow \left.\begin{array}{l} \boxed{AA^T\underline{q} - \underline{b} = 0} \\ \boxed{\underline{x} = A^T \underline{q}} \end{array}\right\} \text{soln}$$

$$\boxed{\begin{array}{c} AA^T\underline{q} - \underline{b} = 0 \\ \underline{x} = A^T \underline{q} \end{array}}$$