## Open-loop control

minimize $PI = \int_a^b \mathcal{L}(\underline{x}, \underline{u})\, dt$

subject to the constraint

$$\underline{0} = \underline{f}(\underline{x}, \underline{u}) - \dot{\underline{x}}$$

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Solution 1760 method of Lagrange multipliers

### single statement

$$\text{minimize } V = \int_a^b \left[ \mathcal{L} + \underline{\lambda}^T \left\{ \underline{f} - \dot{\underline{x}} \right\} \right] dt$$

(* $\underline{\lambda}$ and $\dot{\underline{x}}$)

where $\underline{\lambda}$ = Lag. multi vector ie $\underline{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_N \end{bmatrix}$

and

$\underline{\lambda}^T = [\lambda_1 \ \lambda_2 \ .. \ \lambda_N]$ transpose of $\underline{\lambda}$

### Consider the term*

$$\int_a^b \underline{\lambda}^T \dot{\underline{x}}\, dt = \underline{\lambda}^T \underline{x} \Big|_a^b - \int_a^b \dot{\underline{\lambda}}^T \underline{x}\, dt$$

## modified statement

$$\min. \quad V = \int_a^b \left\{ \left[ \ell + \underline{f}^T \underline{f} \right] + \dot{\underline{\lambda}}^T \underline{x} \right\} dt \; - \underline{\lambda}^T \underline{x} \Big|_a^b$$

boundary cond.

## unknowns

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \quad \underline{\lambda} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{bmatrix}$$

$$\underline{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_M \end{bmatrix}$$

## example

Find $V$ given

$$PI = \int_a^b \left[ x_1^2 + 2x_2^2 + u^2 \right] dt$$

const. $0 = u + a x_1 + b x_2 - \dot{x}_1$

$0 = 6u + c x_1 + d x_2 - \dot{x}_2$

parts $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\underline{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$ $\underline{\lambda}^T = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix}$

$$V = \int_a^b \left( \underbrace{\ell + \underline{\lambda}^T \underline{f}}_{1^\circ} + \underbrace{\dot{\underline{\lambda}}^T \underline{x}}_{2^\circ} \right) dt - \underline{\lambda}^T \underline{x} \Big|_a^b$$

$1^\circ$ $\ell + \underline{\lambda}^T \underline{f} = x_1^2 + 2x_2^2 + u^2 + \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} u + a x_1 + b x_2 \\ 6u + c x_1 + d x_2 \end{bmatrix}$

$2^\circ$ $\dot{\underline{\lambda}}^T \underline{x} = \begin{bmatrix} \dot{\lambda}_1 & \dot{\lambda}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \dot{\lambda}_1 x_1 + \dot{\lambda}_2 x_2$

$3^\circ$ $\underline{\lambda}^T \underline{x} \Big|_a^b = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda_1 x_1 + \lambda_2 x_2$

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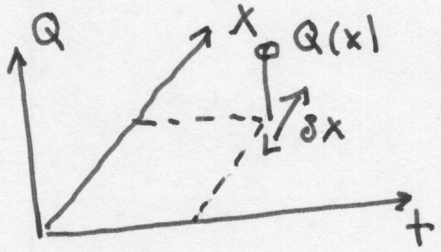
$$\ell + \underline{\lambda}^T \underline{f} = x_1^2 + 2x_2^2 + u^2 + \lambda_1 \left[ u + a x_1 + b x_2 \right] + \lambda_2 \left[ 6u + c x_1 + d x_2 \right]$$

## Minimizing V

Consider the functional $Q[x(t)]$




interest in $Q(x+\delta x)$

To find min of V take the 1st increment of $Q(x)$ and set the result equal to zero

$$\delta Q = \left.\frac{\partial Q}{\partial x}\right|_{\substack{\delta x = 0 \\ t \text{ fixed}}}$$

Soln: $\delta Q = 0$ $\begin{cases} \delta x = 0 \\ \frac{\partial Q}{\partial x} = 0 \end{cases}$ implies x is fixed

example

$$Q(\underline{x}) \quad \text{where} \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find $\delta Q$

$$\delta Q = \frac{\partial Q}{\partial x_1}\delta x_1 + \frac{\partial Q}{\partial x_2}\delta x_2$$

$$\delta Q = \begin{bmatrix} \frac{\partial Q}{\partial x_1} & \frac{\partial Q}{\partial x_2} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$$

$$\delta Q = \left[\frac{\partial Q}{\partial \underline{x}}\right]^T \delta \underline{x} \qquad \text{where } \frac{\partial Q}{\partial \underline{x}} = \begin{bmatrix} \frac{\partial Q}{\partial x} \\ \frac{\partial Q}{\partial x_2} \end{bmatrix}$$

$$\delta \underline{x} = \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$$

Return

$$V = \int_a^b \left\{ \underbrace{\left[ \ell + \underline{\lambda}^T \underline{f} \right]}_{\equiv H} + \dot{\underline{\lambda}}^T \underline{x} \right\} dt - \underline{\lambda}^T \underline{x} \Big|_a^b$$

SFP (state function of Pontryagin

$$V = \int_a^b \left\{ H + \dot{\underline{\lambda}}^T \underline{x} \right\} - \underline{\lambda}^T \underline{x} \Big|_a^b$$

$$\delta V = \int_a^b \delta \left\{ H + \dot{\underline{\lambda}}^T \underline{x} \right\} dt - \delta \left( \underline{\lambda}^T \underline{x} \right) \Big|_a^b$$

note: $H(\underline{x}, \underline{u})$

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$$\delta \left( H + \dot{\underline{\lambda}}^T \underline{x} \right) = 0 \qquad \left[ \frac{\partial H}{\partial \underline{x}} + \dot{\underline{\lambda}} \right]^T \delta \underline{x} + \left[ \frac{\partial H}{\partial \underline{u}} \right]^T \delta \underline{u} = 0$$

$$\delta \left( \underline{\lambda}^T \underline{x} \right) \Big|_a^b = 0 \Rightarrow \underline{\lambda}^T \delta \underline{x} \Big|_a = 0 \Rightarrow \begin{cases} \underline{\lambda}^T \big|_a = 0 & \underline{x} \text{ free} \\ \delta \underline{x} \big|_a = 0 & \underline{x} \text{ fixed} \end{cases}$$

$$\underline{\lambda}^T \delta \underline{x} \Big|_b = 0 \Rightarrow \begin{cases} \underline{\lambda}^T \big|_b = 0 & \text{free} \\ \delta \underline{x} \big|_a = 0 & \text{fixed} \end{cases}$$

## Summary

$$-\dot{\underline{\lambda}} = \frac{\partial H}{\partial \underline{x}} \quad \longrightarrow N \text{ eqn}$$

$$\frac{\partial H}{\partial \underline{u}} = 0 \quad \longrightarrow m \text{ eqn}$$

$$\dot{\underline{x}} = \underline{f} \quad \longrightarrow N \text{ eqn}$$

with BC:

$$\underline{\lambda}^T \delta \underline{x} \big|_a = 0 \rightarrow N \text{ bc}$$

$$\underline{\lambda}^T \delta \underline{x} \big|_b = 0 \rightarrow N \text{ bc}$$

and $H = \ell + \underline{\lambda}^T \underline{f}$